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NUMERICALLY SOLVING A TRANSIENT HEAT CONDUCTION PROBLEM WITH CONVECTION AND RADIATION

by

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June 1993

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Numerically Solving A Transient Heat Conduction Problem With Convection and Radiation

by

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ABSTRACT

The transient surface temperature distribution is determined for the flat plate and sphere subjected to cooling by combined convection and radiation. In the study, the initial boundary value problem is reduced to a singular nonlinear Volterra integral equation of the second kind using the integral transform method. Several numerical techniques are introduced in an attempt to find an approximate solution of the problem: The method of successive approximations, the Runge-Kutta method, and the finite difference method. The integral equation is solved numerically by the Runge-Kutta method of orders 1, 3, and 5. In addition, the finite difference method is implemented to solve the initial boundary value problem, and the solutions are compared with those generated by the Runge-Kutta method. All the numerical results are presented graphically. Limitations and difficulties involved in these schemes are discussed. At the end, a numerical algorithm for solving the problem is proposed.

TABLE OF CONTENTS

I. ANALYTIC SOLUTIONS OF THE HEAT EQUATION SUBJECT TO CONVECTIVE AND RADIATIVE BOUNDARY CONDITIONS

A. INTRODUCTION

During the 60's, space technology advanced so much that the research of the temperature behaviour of bodies exposed to a deep space environment became crucial. In particular, transient heat or cooling of solids of different shapes by convection and thermal radiation was becoming highly important in many engineering applications. An example of these applications is the temperature distributions of rocket motors. An extensive investigation of the problem has been conducted and a lot of literature on the subject was published during the 60's and 70's. A detailed review of most of these papers is not intended here; instead a brief summary of the major ones will be given.

As early as 1962, Fairall, et.al.[6] generated a numerical solution for the problem using an explicit finite difference scheme; this paper served as pioneer work in the area of the research. Later, various finite difference schemes were devised to deal with the nonlinear boundary condition. The main difficulty in these schemes is the appearance of severe oscillations in the determined temperature values for high heat flux situations. Von Rosenberg [10] proposed a hybrid of an iterative technique and implicit finite difference schemes to deal with the nonlinear boundary condition. On the other hand, Crosbie and Viskanta [3,4] transformed the governing equations into a nonlinear Volterra integral equation of the second kind and applied the method of successive approximations to solve the integral equation. Milton and Goss [8,9] developed some heuristic stability criteria for explicit finite difference schemes with nonlinear boundary conditions. It turns out that a very restrictive time step is required for numerical stability which may result in requiring a prohibitive amount of computer time to calculate the long time evolution of the solutions. Williams and Curry [12] surveyed several methods for treating the nonlinear boundary condition in implicit schemes and compared their accuracy and efficiency.

Nonlinearity is commonplace in natural phenomena. Unfortunately, a nonlinear problem often doesn't lend itself to a closed form solution. The problem of transient heat-conduction in a solid becomes nonlinear when the surface of the body is subjected to thermal radiation. When energy transfers through the wall of a body, two cases arise: convection and thermal radiation. The convective heat transfer describes the situation where heat is dissipated according to Newton's Law of Cooling, which states that the rate at which heat is transferred from the body to a surrounding is proportional to the difference in temperature between the body and the environment. The boundary condition that describes convection is nonlinear except for the case where the heat-transfer coefficient is independent of surface temperature, which is technically called forced convection. The radiative heat transfer is based on the Stefan-Boltzmann Law, which states that the heat flux is proportional to the difference between the surface temperature to the fourth power and the source temperature. Pure radiation or pure convection occur whenever one mode of energy transfer predominates over the other.

It is the purpose of this thesis to consider the one-dimensional transient heat conduction problem resulting from a combined convective and radiative heat flux with the objective of determining the surface temperature fields using the numerical methods which are discussed in this study. Another purpose of this thesis is to explore the limitations and difficulties involved in these schemes. References to the work done in similar areas are presented to allow the reader further investigation.

Analytic solutions are derived in one dimension. However, the resulting solutions are not in closed form, and thus impractical to use. Hence, numerical techniques will be studied and employed in the computer in an attempt to find an approximate solution. Numerical results, found by implementing some of the numerical methods discussed below, will be presented and compared. In the conclusion, a numerical scheme is proposed as an alternative to the existing methods. It is open to the readers for justification.

Sections 1(C) and 1(D) describe the derivation of the integral representations of the one dimensional transient heat conduction problem subjected to a combined convective and radiative boundary condition in a rectangular coordinate system. Two integral transform methods, namely the Laplace transform and the eigenvalue expansion, are presented. Observation and comparison are made for the integral equations to yield some useful information about the solutions.

In Chapters II and III, numerical methods for the solutions of the nonlinear Volterra integral equations of the second kind are described. In particular, the method of successive approximations and the Runge-Kutta method are outlined in detail. A brief remark is given for their advantages and limitations in finding solutions to the integral equation.

Chapter IV describes a numerical method which is directly applied to the governing partial differential equation. The technique is called the finite difference method. It is basically a hybrid of finite difference techniques and an iterative scheme proposed. A suggestion is made for the improvement of the algorithm.

In Chapter V, numerical results produced by some of the discussed numerical schemes are presented. The implementation of various methods gave a practical sense of their advantages and limitations. Graphs and tables are set up in such a way that a comparison can be made.

In the next section, a statement of the problem is given. In the statement, the basic assumptions, the governing equation and the boundary-initial conditions are included.

B. STATEMENT OF THE PROBLEM FOR OBTAINING THE SURFACE TEMPERATURE

Considering the one-dimensional, transient, conduction heat transfer problem with combined convection and radiation at its surface, the following assumptions have been made:

1. One-dimensional heat transfer to a solid of a finite length.
2. The solid medium is pure, isotropic, homogeneous, and opaque to thermal radiation.
3. All thermodynamic and transport properties are independent of temperature.
4. The solid does not contain any heat sources or sinks.
5. The fluid is transparent to thermal radiation.
6. The fluid temperature and the ambient temperature are constant.

The non-dimensional form of the governing partial differential equation for the temperature U(x,t) and the appropriate initial boundary conditions are

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}, \qquad 0<x<1, \qquad t>0; \tag{1.1}$$

with initial condition

$$U(x,0) = g(x) \tag{1.2a}$$

and boundary conditions

$$(i) \quad \alpha_1 \frac{\partial U(0,t)}{\partial x} - \alpha_2 U(0,t) = 0 \tag{1.2b}$$

$$(ii) \quad \frac{\partial U(1,t)}{\partial x} - \alpha_3 U(1,t) = -hU^4(1,t). \tag{1.2c}$$

Note: α_1 and α_2 can be any real number, except both cannot be zero at the same time. α_3 is a non-zero real number, and h is a positive real number.

The next section will deal with solving the partial differential equation (1.1) with initial and boundary conditions (1.2a-c) by the Laplace transform method and the eigenvalue expansion method. As an illustration, two special cases with specific values of α_1, α_2, α_3, and h will be considered, and the analytic solutions of these cases at the surface will be derived. It will be shown that the surface temperature satisfies a singular Volterra integral equation of the second kind. At the end of the chapter, we will present the solutions and indicate some useful information about the integral equations.

C. THE LAPLACE TRANSFORM METHOD

In this section, the Laplace transform of equation (1.1) with associated boundary conditions (1.2b,c) is first obtained with respect to time. The resulting boundary value problem is in terms of the Laplace transform of the required solution. Next, the equations are solved for the transformed temperature, and the solution of the stated problem can be found by taking the inverse Laplace transform of the transformed solution. From experience, it can be expected, the Laplace inversion is of some difficulty. To simplify the situation, specific values of α_1, α_2, α_3, and h are considered so that the inverse process is practical without loss of generality. It should be noted that there does exist an inverse Laplace transform for other cases of a more general nature.

Now, define the transform of the temperature function, U(x,t), with respect to time as follows

$$\mathcal{L}[U(x,t)](s) = \int_0^\infty u(x,t)e^{-st}dt = U(x,s). \tag{1.3}$$

After the transformation, the temperature function becomes a function not only of x but also of the parameter s. Assuming that the derivatives with respect to x pass through the transform (differentiation can be accomplished before integration), we have

$$\mathcal{L}[\frac{\partial u(x,t)}{\partial x}](s) = \int_0^\infty \frac{\partial u(x,t)}{\partial x} e^{-st}dt = \frac{\partial U(x,s)}{\partial x} \tag{1.4}$$

$$\mathcal{L}[\frac{\partial^2 u(x,t)}{\partial x^2}](s) = \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} e^{-st}dt = \frac{\partial^2 U(x,s)}{\partial x^2} \tag{1.5}$$

The rule for transforming a derivative with respect to time can be found using integration by parts. Thus, the Laplace transform of the derivatives of U(x,t) with respect to the transformed variable t is given by

$$\mathcal{L}[\frac{\partial u(x,t)}{\partial t}](s) = \int_0^\infty \frac{\partial u(x,t)}{\partial t} e^{-st}dt = sU(x,s) - U(x,0). \tag{1.6}$$

Now, applying the Laplace transform to the initial-boundary value problem (1.1), (1.2a-c) we remove all time derivatives. Holding s fixed, we have the following ordinary differential equation in x

$$\frac{d^2 U(x,s)}{dx^2} - sU(x,s) = -g(x), \qquad 0<x<1 \tag{1.7}$$

with boundary conditions

$$\alpha_1 \frac{dU(0,s)}{dx} - \alpha_2 U(0,s) = 0, \ for\ x = 0 \tag{1.8a}$$

$$\frac{dU(1,s)}{dx} - \alpha_3 U(1,s) = -h\mathcal{L}[U^4(1,s)], \ for \ x=1 \tag{1.8b}$$

Notice that the initial condition, g(x), is incorporated in the ordinary differential equation. In order to solve (1.7) and (1.8a,b), we must first solve for the general solution of the corresponding homogeneous differential equation and a particular solution of (1.7) satisfying (1.8a,b). Now, consider the general solution of the homogeneous equation for (1.7),

$$U_{hom}(x,s) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}, \tag{1.9}$$

where

$$\lambda_{1,2} = \pm\sqrt{s} \tag{1.10}$$

which are given by the roots of the auxiliary equation

$$\lambda^2 - s = 0. \tag{1.11}$$

In the following paragraph we employ the method of variation of parameters to solve for a particular solution of (1.7).

Let

$$U_p(x,s) = U_1 v_1(x,s) + U_2 v_2(x,s), \tag{1.12}$$

be a particular solution where $U_1(x,s)$ and $U_2(x,s)$ are any two linearly independent solutions of the corresponding homogeneous equation. In this case, choose $U_1(x,s) = e^{\sqrt{s}x}$ and $U_2(x,s) = e^{-\sqrt{s}x}$. The object here is to find $v_1(x,s)$ and $v_2(x,s)$ such that the following equations are satisfied

$$e^{\sqrt{s}x} v_1'(x,s) + e^{-\sqrt{s}x} v_2'(x,s) = 0, \tag{1.13}$$

$$\sqrt{s}\, e^{\sqrt{s}\,x} v_1'(x,s) - \sqrt{s}\, e^{-\sqrt{s}\,x} v_2'(x,s) =- g(x) \tag{1.14}$$

By Cramer's rule,

$$v_1'(x,s) = \frac{g(x)\, e^{-\sqrt{s}x}}{-2\sqrt{s}} \tag{1.15}$$

and

$$v_2'(x,s) = \frac{g(x)\, e^{\sqrt{s}x}}{2\sqrt{s}}. \tag{1.16}$$

By integrating (1.15) and (1.16), we obtain

$$v_1(x,s) = -\int_0^x \frac{g(z)\, e^{-\sqrt{s}z} dz}{2\sqrt{s}} + v_1(0,s) \tag{1.17}$$

$$v_2(x,s) = \int_0^x \frac{g(z)\,e^{\sqrt{s}z}dz}{2\sqrt{s}} - v_2(0,s) \tag{1.18}$$

Thus, the general solution to (1.7) and (1.8a,b) is

$$U(x,s) = U_h(x,s) + U_p(x,s) \tag{1.19}$$

that is,

$$U(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + e^{\sqrt{s}x}v_1(x,s) + e^{-\sqrt{s}x}v_2(x,s)\,, \tag{1.20}$$

where A and B are arbitrary constants and $v_1(x,s)$, $v_2(x,s)$ are given by (1.17) and (1.18), respectively. To determine A and B, boundary conditions (1.8a,b) are used along with the following procedure. The derivative of U(x,s) from equation (1.20) is found to be

$$\frac{dU(x,s)}{dx} = A\sqrt{s}e^{\sqrt{s}x} - B\sqrt{s}e^{-\sqrt{s}x} + \sqrt{s}e^{\sqrt{s}x}v_1(x,s) +$$

$$e^{\sqrt{s}x}v_1'(x,s) + e^{-\sqrt{s}x}v_2'(x,s) - \sqrt{s}e^{-\sqrt{s}x}v_2(x,s)\,. \tag{1.21}$$

Let x = 0. Then (1.15) and (1.16) give

$$v_1'(0,s) + v_2'(0,s) = 0\,.$$

(1.18a), (1.20), and (1.21) then imply

$$\alpha_1[A\sqrt{s} - B\sqrt{s} + \sqrt{s}v_1(0,s) - \sqrt{s}v_2(0,s)]$$

$$- \alpha_2[A + B + v_1(0,s) + v_2(0,s)] = 0 \qquad (1.22)$$

By rearranging the terms, (1.22) becomes

$$A(\alpha_1\sqrt{s} - \alpha_2) - B(\alpha_1\sqrt{s} + \alpha_2) =$$

$$(\alpha_1\sqrt{s} + \alpha_2)v_2(0,s) - (\alpha_1\sqrt{s} - \alpha_2)v_1(0,s). \qquad (1.23)$$

Similarly let x = 1. Then (1.15) and (1.16) give

$$e^{\sqrt{s}}v_1'(1,s) + e^{-\sqrt{s}}v_2'(1,s) = 0 . \qquad (1.24)$$

Therefore (1.8a), (1.20), and (1.21) then imply

$$A\sqrt{s}e^{\sqrt{s}} - B\sqrt{s}e^{-\sqrt{s}} + \sqrt{s}e^{\sqrt{s}}v_1(1,s) - \sqrt{s}e^{-\sqrt{s}}v_2(1,s) -$$

$$\alpha_3[Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + e^{\sqrt{s}}v_1(1,s) + e^{-\sqrt{s}}v_2(1,s)] =$$

$$- h\mathcal{L}[U^4(1,t)] . \qquad (1.25)$$

By a similar manipulation of the terms, (1.25) becomes

$$A(\sqrt{s}e^{\sqrt{s}} - \alpha_3e^{\sqrt{s}}) - B(\sqrt{s}e^{-\sqrt{s}} + \alpha_3e^{-\sqrt{s}}) =$$

$$(\sqrt{s}e^{-\sqrt{s}} + \alpha_3e^{-\sqrt{s}})v_2(1,s) - (\sqrt{s}e^{\sqrt{s}} - \alpha_3e^{\sqrt{s}})v_1(1,s) -$$

$$h\mathcal{L}[U^4(1,t)] . \qquad (1.26)$$

Equations (1.23) and (1.26) form a system of two equations in the two unknowns A and B. By Cramer's rule, A and B are as follows

$$A = \frac{num1}{der} \tag{1.27}$$

where

$$num1 =$$

$$\{(\alpha_1\sqrt{s}-\alpha_2)v_1(0,s)+(\alpha_1\sqrt{s}+\alpha_2)[v_2(1,s)-v_2(0,s)]\}(\sqrt{s}e^{-\sqrt{s}}+\alpha_3 e^{-\sqrt{s}})$$

$$-[(\alpha_1\sqrt{s}+\alpha_2)v_1(1,s)](\sqrt{s}e^{\sqrt{s}}-\alpha_3 e^{\sqrt{s}}) - h\mathcal{L}[U^4(1,t)](\alpha_1\sqrt{s}+\alpha_2) \quad ,$$

and

$$den =$$

$$(\alpha_2 - \alpha_3\alpha_1)\sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}}) + (\alpha_1 s - \alpha_3\alpha_2)(e^{\sqrt{s}}-e^{-\sqrt{s}}) \quad .$$

$$B = \frac{num2}{den} \tag{1.28}$$

where

$$num2 =$$

$$\{(\alpha_1\sqrt{s}-\alpha_2)[v_1(0,s)-v_1(1,s)]-(\alpha_1\sqrt{s}+\alpha_2)v_2(0,s)\}(\sqrt{s}e^{\sqrt{s}}-\alpha_3 e^{\sqrt{s}})$$

$$-h\mathcal{L}[U^4(1,t)](\alpha_1\sqrt{s}-\alpha_2)+[(\alpha_1\sqrt{s}-\alpha_2)v_2(1,s)](\sqrt{s}e^{-\sqrt{s}}+\alpha_3 e^{-\sqrt{s}})$$

and den is the same.

Thus, the general solution of (1.7), (1.8a,b) is given by (1.20) where A, B, $\upsilon_1(x,s)$, and $\upsilon_2(x,s)$ are given by (1.27), (1.28), (1.17), and (1.18), respectively. Theoretically, the analytic solution of partial differential equation (1.1) with initial and boundary conditions (1.2a,b,c) can be obtained by taking the Laplace inversion of U(x,s), and thus, the surface solution can be found by putting x equal to one in U(x,s). In practice, however, the inverse Laplace transformation process is highly unstable in that singularities may exist. Also, the transforms are difficult to find. In the next paragraph below we consider two special cases where the inversion is feasible. In each case, values for the parameters correspond to a specific geometrical configuration of a body.

<u>Case 1:</u> $\alpha_1 = 1,\ \alpha_2 = 0,\ \alpha_3 = -1,\ h = 1$

This set of values corresponds to a "flat plate" with a given initial temperature and which is being heated or cooled by combined convection and radiation. The term "flat plate" is taken here to mean a solid slab of finite thickness which is bounded by a pair of vertical lines at ± ½ thus having a width of 1. Substituting the given values for the parameters in (1.20) we obtain the transformed surface temperature

$$U(1,s) = Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + e^{\sqrt{s}}v_1(1,s) + e^{-\sqrt{s}}v_2(1,s), \qquad (1.29)$$

where,

$$A = \frac{[\sqrt{s}v_1(0,s) + \sqrt{s}v_2(1,s) - \sqrt{s}v_2(0,s)]\,(\sqrt{s}e^{-\sqrt{s}} - e^{-\sqrt{s}})}{\sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}}) + s(e^{\sqrt{s}} - e^{-\sqrt{s}})}$$

$$+ \frac{-\,[\sqrt{s}v_1(1,s)]\,(\sqrt{s}e^{\sqrt{s}} + e^{\sqrt{s}}) - h\mathcal{L}[U^4(1,t)]\sqrt{s}}{\sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}}) + s(e^{\sqrt{s}} - e^{-\sqrt{s}})} \qquad (1.30)$$

$$B = \frac{[\sqrt{s}v_1(0,s) - \sqrt{s}v_1(1,s) - \sqrt{s}v_2(0,s)]\,(\sqrt{s}e^{\sqrt{s}} + e^{\sqrt{s}})}{(\alpha_2 - \alpha_3\alpha_1)\sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}}) + (\alpha_1 s - \alpha_3\alpha_2)\,(e^{\sqrt{s}} - e^{-\sqrt{s}})}$$

$$+ \frac{-\,h\mathcal{L}[U^4(1,t)]\sqrt{s} + [\sqrt{s}v_2(1,s)]\,(\sqrt{s}e^{-\sqrt{s}} - e^{-\sqrt{s}})}{(\alpha_2 - \alpha_3\alpha_1)\sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}}) + (\alpha_1 s - \alpha_3\alpha_2)\,(e^{\sqrt{s}} - e^{-\sqrt{s}})} \qquad (1.31)$$

$$v_1(1,s) = -\int_0^1 \frac{g(x')\,e^{-\sqrt{s}x'}}{2\sqrt{s}}dx' + v_1(0,s) \qquad (1.32)$$

and

$$v_2(1,s) = \int_0^1 \frac{g(x')\,e^{\sqrt{s}x'}}{2\sqrt{s}}dx' + v_2(0,s) \qquad (1.33)$$

Now substituting (1.30)-(1.33) into (1.29) and simplifying the results gives:

$$U(1,s) = \frac{\int_0^1 g(x')\,e^{\sqrt{s}x'}dx' + \int_0^1 g(x')\,e^{-\sqrt{s}x'}dx'}{(e^{\sqrt{s}} + e^{-\sqrt{s}}) + \sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}$$

$$+ \frac{-\,h\mathcal{L}[U^4(1,t)]\,(e^{\sqrt{s}} + e^{-\sqrt{s}})}{(e^{\sqrt{s}} + e^{-\sqrt{s}}) + \sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}\;. \qquad (1.34)$$

Suppose the initial temperature is 1, that is,

$$g(x) = 1. \tag{1.35}$$

The boundary conditions associated with the given values of the parameters and initial condition (1.35) constitute a cooling process. With (1.35), the transformed surface temperature becomes

$$U(1,s) = \frac{\frac{1}{\sqrt{s}}(e^{\sqrt{s}} - e^{-\sqrt{s}}) - h\,\mathcal{L}[U^4(1,t)]\,(e^{\sqrt{s}} + e^{-\sqrt{s}})}{(e^{\sqrt{s}} + e^{-\sqrt{s}}) + \sqrt{s}(e\sqrt{s} - e^{-\sqrt{s}})}\ . \tag{1.36}$$

If (1.36) is multiplied through by

$$\frac{1}{\sqrt{s}}\left[\frac{e^{\sqrt{s}} + e^{-\sqrt{s}} + \sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}{(e^{\sqrt{s}} - e^{-\sqrt{s}})}\right]$$

and then simplified,

$$U(1,s) = \frac{1}{s} - \frac{\mathcal{L}[h\,U^4(1,t) + U(1,t)]\,(e^{\sqrt{s}} + e^{-\sqrt{s}})}{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})} \tag{1.37}$$

is obtained. Equation (1.37) is ready to be inverted. In order to perform the inversion of (1.37), the following two Laplace transforms have to be computed

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] \quad and \quad \mathcal{L}^{-1}\left[\frac{(e^{\sqrt{s}} + e^{-\sqrt{s}})}{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}\right]\ .$$

In fact the transforms can be found from any standard Laplace transform table. By the convolution theorem, the surface temperature in time t is given by

$$U(1,t) = 1 - \int_0^t \frac{[1 + 2\sum_{k=1}^{\infty} e^{-\frac{k^2}{(t-\tau)}}]}{\sqrt{\pi(t-\tau)}} [U^4(1,\tau) + U(1,\tau)]\,d\tau \qquad (1.38)$$

and by the Poisson summation formula [14], (1.38) can be written as

$$U(1,t) = 1 - \int_0^t [1+2\sum_{k=1}^{\infty} e^{-k^2\pi^2(t-\tau)}]\,[U^4(1,\tau)+U(1,\tau)]\;d\tau\;. \quad (1.39)$$

Hence, the problem of transient cooling of a flat plate by combined convection and thermal radiation has been reduced to solving a nonlinear Volterra integral equation of the second kind.

Case 2: $\alpha_1 = 0,\ \alpha_2 = -1,\ \alpha_3 = 1,\ h = 1$

This set of values corresponds to the case where a spherical body of radius 1 with a given initial temperature is being heated or cooled by combined convection and radiation. Since the procedures used to solve the problem are basically those described in case 1, the mathematical details will be omitted and only the main steps will be presented. Consider equation (1.20), the general solution of the boundary value problem. The given values for the parameters are first substituted into (1.17), (1.18), (1.27), and (1.28). Then, (1.20) is simplified as in the previous case. After a tedious calculation, the transformed surface temperature is given by

$$U(1,s) = \frac{\int_0^1 g(x')\, e^{-\sqrt{s}x'}dx' - \int_0^1 g(x')\, e^{\sqrt{s}x'}dx' +}{(e^{\sqrt{s}} - e^{-\sqrt{s}}) - \sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}})}$$

$$+ \frac{h\, \mathcal{L}[U^4(1,t)]\, (e^{\sqrt{s}} + e^{-\sqrt{s}})}{(e^{\sqrt{s}} - e^{-\sqrt{s}}) - \sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}})}. \quad (1.40)$$

Suppose the initial temperature is chosen to be

$$g(x) = x. \quad (1.41)$$

Boundary conditions associated with the given values of the parameters and initial condition (1.41) again constitute a cooling process. With (1.41), the transformed surface temperature becomes

$$U(1,s) = \frac{1}{s} - \frac{h\, \mathcal{L}[U^4(1,t)]\, (e^{\sqrt{s}} - e^{-\sqrt{s}})}{\sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}}) - (e^{\sqrt{s}} - e^{-\sqrt{s}})} \quad (1.42)$$

which is now ready to be inverted. In order to perform the Laplace inversion of (1.42) the following two inverse Laplace transforms need to be computed

$$\mathcal{L}^{-1}[\frac{1}{s}] \quad and \quad \mathcal{L}^{-1}[\frac{(e^{\sqrt{s}} - e^{-\sqrt{s}})}{\sqrt{s}(e^{\sqrt{s}} + e^{-\sqrt{s}}) - (e^{\sqrt{s}} - e^{-\sqrt{s}})}] .$$

The first inverse Laplace transform is obvious. However, the second one is not so obvious. Details of the derivation of the second inverse Laplace transform are given in [1]. The surface temperature in time t, obtained by inverting (1.42), is

$$U(1,t) = 1 - \int_0^t [3 + 2\sum_{k=1}^{\infty} e^{-\beta_k^2(t-\tau)}]\, U^4(1,\tau)\, d\tau, \tag{1.43}$$

where β_k is the $k^{\underline{th}}$ positive root of the transcendental equation

$$\beta_k = \tan\,\beta_k \quad . \tag{1.44}$$

Hence, the problem of transient cooling of a sphere by combined convection and thermal radiation has been again reduced to solving a nonlinear Volterra integral equation of the second kind. As we have mentioned above, one of the drawbacks of the Laplace transform method is that there are only a few cases in which the transformed solution can be practically inverted into the required solution. In the next section, the eigenvalue expansion method is introduced as an alternative to the above method. One may find the eigenvalue method more practical for solving for the analytic solution of the heat equation with nonlinear boundary conditions.

D. THE EIGENVALUE EXPANSION METHOD

The fundamental idea of the eigenvalue expansion method is to transform the given boundary value problem by the eigenfunctions obtained from the associated eigenfunction problem. By the completeness theorem (which states that any piecewise smooth function can be represented by a generalized series of eigenfunctions) we can show that separation of variables, i.e., $u(x,t) = X(x)T(t)$, may lead to the solution of the problem expressed as an infinite sum of the eigenfunctions with appropriate coefficients determined by the orthogonality property of eigenfunctions. Applying these procedures to the partial differential equation (1.1) and initial boundary conditions (1.2a,b,c) yields the following main results

$$\frac{d^2X(x)}{dx^2} + \beta^2 X(x) = 0, \qquad 0 < x < 1 \tag{1.45}$$

with boundary conditions

$$\alpha_1 \frac{dX(0)}{dx} - \alpha_2 X(0) = 0, \tag{1.46}$$

and

$$\frac{dX(1)}{dx} - \alpha_3 X(1) = 0 \ . \tag{1.47}$$

Parameters α_1 and α_2 can be any real number except they cannot be zero at the same time. α_3 is a non-zero real number. According to the theory of ordinary differential equations, the general solution of (1.45) is

$$X(x) = c_1\cos(\beta x) + c_2\sin(\beta x) \ . \tag{1.48}$$

Applying boundary conditions (1.46) and (1.47) to equation (1.48) gives the following system of equations

$$\alpha_1\beta c_2 = \alpha_2 c_1 \tag{1.49}$$

$$(c_2\beta - c_1\alpha_3)\cos\beta = (c_1\beta + c_2\alpha_3)\sin\beta \ . \tag{1.50}$$

Note that boundary value problem (1.45 - 1.47) is in the class of Sturm-Liouville problems for which all eigenvalues are real and the eigenfunctions corresponding to different eigenvalues are orthogonal. Thus, if the parameters in (1.49) and (1.50) are specified, there will exist eigenvalues, β_n, where n = 1,2,..., and the corresponding eigenfunctions, $X_n(x)$, such that the temperature function, U(x,t), can be expanded in a Fourier expansion of the form

$$U(x,t) = \sum_{n=1}^{\infty} U_n(t)\, X_n(x) \ , \tag{1.51}$$

where the Fourier coefficients, $U_n(t)$, are given by

$$U_n(t) = \int_0^1 U(x,t)\ X_n(x)\, dx \ . \tag{1.52}$$

Now, taking the finite Fourier integral transform of the heat equation (1.1) with respect to $X_n(x)$ gives

$$\frac{d}{dt}\int_0^1 U(x,t)\ X_n(x)\ dx = \int_0^1 \frac{\partial^2 U}{\partial x^2} X_n(x)\ dx\ . \tag{1.53}$$

Performing integration by parts of the right hand expression in equation (1.53) and substituting in (1.52) yields the following ordinary differential equation for $U_n(t)$

$$\begin{aligned}\frac{dU_n(t)}{dt} &= \frac{\partial U(1,t)}{\partial x} X_n(1) - \frac{\partial U(0,t)}{\partial x} X_n(0) - U(1,t) X_n'(1) \\ &\quad + U(0,t) X_n'(0) + \int_0^1 U(x,t) X_n''(x)\ dx\ . \end{aligned} \tag{1.54}$$

With boundary conditions (1.46) and (1.47), the right hand side of (1.54) can be simplified. Then, by the integrating factor method, the solution of equation (1.54) can be obtained. Hence, the resulting integral equation for U(x,t) takes the form of (1.51) with $U_n(t)$ solved in (1.54). Lastly, by putting x = 1, a nonlinear Volterra integral equation of the second kind for the surface temperature U(1,t) is obtained.

As in the previous section, the integral equation for the surface temperature will be explicitly determined for two special cases: the flat plate and the sphere. Details of the derivation of the solution will be produced in the case of the flat plate, but only major results will be given in the case of the sphere.

<u>Case 1:</u> $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = -1$, $h = 1$

As mentioned in section 1(C), this set of parameters corresponds to the geometrical configuration of a flat plate. Substituting the values of α_1, α_2, and α_3 in (1.49) and (1.50), c_1 equals zero, and (1.50) leads to

$$\cos\beta_n = \beta_n \sin\beta_n \quad \Rightarrow \quad \frac{1}{\beta_n} = \tan\beta_n \,, \tag{1.55}$$

where $\cos \beta_n \neq 0$.

So, the family of orthogonal eigenfunctions are

$$X_n(x) = \cos(\beta_n x) \,, \tag{1.56}$$

where $n = 1,2,3,\ldots,$ and $\{\beta_n\}_{n=1}^{\infty}$ is the set of distinct eigenvalues which are the roots of (1.55) with the property

$$0<\beta_1<\beta_2<\beta_3<\ldots \quad .$$

Next, applying the finite Fourier integral transform of the heat equation (1.1) yields (1.54) in terms of $X_n(x)$. Using the boundary conditions

$$\frac{\partial U(0,t)}{\partial x} = 0 \,, \tag{1.57}$$

$$\frac{\partial U(1,t)}{\partial x} + U(1,t) = -h\, U^4(1,t)\ , \tag{1.58}$$

$$X'(1) + X(1) = 0\ , \tag{1.59}$$

$$X'(0) = 0\ , \tag{1.60}$$

and the fact that

$$X_n''(x) = -\beta_n^2 X_n(x) \tag{1.61}$$

produces the following ordinary differential equation for $U_n(t)$

$$\frac{dU_n(t)}{dt} + \beta_n^2 U_n(t) = -h\, X_n(1)\, U^4(1,t)\ . \tag{1.62}$$

Note that (1.62) is a first order linear ordinary differential equation. We find the solution to be

$$U_n(t) = U_n(0)\, e^{-\beta_n^2 t} - h\, X_n(1) \int_0^t e^{-\beta_n^2 (t-\tau)} U^4(1,\tau)\ d\tau\ , \tag{1.63}$$

where

$$U_n(0) = \int_0^1 g(x)\, X_n(x)\ dx\ . \tag{1.64}$$

Thus, with h = 1, the integral equation for U(x,t) takes the form

$$U(x,t) = \sum_{n=1}^{\infty} \{ \frac{[U_n(0)\, e^{-\beta_n^2 t} - (1)\ X_n(1) \int_0^t e^{-\beta_n^2 (t-\tau)} U^4(1,\tau)\ d\tau]\, x_n(x)}{\int_0^1 X_n^2(x)\, dx} \}$$

where $U_n(0)$ and $X_n(x)$ are defined by (1.64) and (1.56), respectively. Lastly, by putting x = 1, the integral solution for the surface temperature U(1,t) is determined to be

$$U(1,t) = \sum_{n=1}^{\infty} \{ \frac{e^{-\beta_n^2 t} X_n(1) \int_0^1 g(x)\, X_n(x)\ dx}{\int_0^1 X_n^2(x)\ dx} - \frac{\int_0^t X_n^2(1)\, e^{-\beta_n^2 (t-\tau)} U^4(1,\tau)\ d\tau}{\int_0^1 X_n^2(x)\ dx} \}, \qquad (1.65)$$

where g(x) is the initial condition, and $X_n(x)$, and β_n are defined as above.

<u>Case 2:</u> $\alpha_1 = 0$, $\alpha_2 = -1$, $\alpha_3 = 1$, h = 1

In this case, a spherical body is considered. In a similar fashion, the family of orthogonal eigenfunctions can be found and are given by

$$X_n(x) = \sin(\beta_n x)\ , \qquad (1.66)$$

where n = 1,2,3,..., and β_n is the set of distinct eigenvalues that are the roots of

$$\beta_n = \tan \beta_n \tag{1.67}$$

with the property

$$0 < \beta_1 < \beta_2 < \beta_3 < \dots \quad .$$

After applying the finite Fourier integral transform of heat equation (1.1) with respect to $X_n(x)$, the following ordinary differential equation for $U_n(t)$ is obtained

$$\frac{dU_n(t)}{dt} + \beta_n^2 U_n(t) = -h\, X_n(1)\, U^4(1,t) \ . \tag{1.68}$$

Thus, the solution of equation (1.68) is

$$U_n(t) = U_n(0)\, e^{-\beta_n^2 t} - h\, X_n(1) \int_0^1 e^{-\beta_n^2 (t-\tau)} U^4(1,\tau)\ d\tau \ , \tag{1.69}$$

where

$$U_n(0) = \int_0^1 g(x)\, X_n(x)\, dx \ . \tag{1.70}$$

So therefore, with h = 1, the integral equation for U(x,t) takes the form

$$U(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{[U_n(0)\, e^{-\beta_n^2 t} - (1)\ X_n(1) \int_0^t e^{-\beta_n^2 (t-\tau)} U^4(1,\tau)\ d\tau]\, x_n(x)}{\int_0^1 X_n^2(x)\, dx} \right\}$$

where $U_n(0)$ and $X_n(x)$ are defined by (1.70) and (1.66), respectively. Lastly, by putting x = 1, the integral equation for the surface temperature U(1,t) becomes

$$U(1,t) = \sum_{n=1}^{\infty} \left\{ \frac{e^{-\beta_n^2 t} X_n(1) \int_0^1 g(x) X_n(x)\ dx}{\int_0^1 X_n^2(x)\ dx} - \frac{\int_0^t X_n^2(1)\, e^{-\beta_n^2 (t-\tau)} U^4(1,\tau)\ d\tau}{\int_0^1 X_n^2(x)\ dx} \right\},$$

where g(x) is the initial condition, and $X_n(x)$ and β_n are defined as above.

E. REMARKS

The solution presented above is not complete in the sense that the surface temperature is only determined for two cases. The solution for other geometrical configurations can be found in some of the literature listed in the references, specifically 3, 5, 6, and 11.

The surface temperature solutions which have been derived above by both methods fall into the form

$$U(1,t) = \phi(t) - h\int_0^t \{a + \sum_{k=1}^{\infty} b_k e^{-c_k^2 (t-\tau)}\} F[U(1,\tau)]\ d\tau\ , \qquad (1.72)$$

where F is a nonlinear function of U(1,t), and c_k, b_k, a, and h are some constants. Equation (1.72) is a nonlinear

Volterra integral equation of the second kind. $\phi(t)$ is a function which is usually called the "lag" part of the integral equation. The integral in (1.72) is often referred to as the "Volterra" part of the integral equation. In addition, the piece within the braces of the Volterra part is called the "kernel" of the integral equation. As these integral equations are being examined, several facts about (1.72) are summarized as follows:

1). All of these integral equations are singular because as τ approaches t, the kernel blows up to infinity.

2). All of the infinite series satisfy the following property:

If $f(t-\tau)$ is used to denote an infinite series, then $\lim_{t\to\infty} f(t)$ = constant, thus remaining finite.

3). The lag part, $\phi(t)$, and the kernel of the integral equations are determined by the geometry of the body considered.

The above "facts" are concluded from the two special cases without loss of generality. In each of the next three chapters, a different numerical method for solving the problem stated in section A will be introduced. Both the method of successive approximations and the Runge-Kutta method are numerical techniques used to deal with the integral representation of the problem, whereas the finite difference method is applied directly to the governing equations.

II. THE METHOD OF SUCCESSIVE APPROXIMATIONS

A. INTRODUCTION

The surface temperature of a body subject to a combined convective and radiative boundary condition, as seen in Chapter I, is given by the solution of a singular nonlinear Volterra integral equation of the second kind. Since the integral equation is not in closed form and is nonlinear, numerical techniques seem to be the most practical way to tackle the problem. Over the past twenty years, a lot of research has been done on the numerical solution of an integral equation of the form

$$U(1,t) = \phi(t) - h\int_0^t \{a + \sum_{k=1}^{\infty} b_k e^{-c_k^2(t-\tau)}\} F[U(1,\tau)]\ d\tau\ . \qquad (2.1)$$

Among the existing numerical methods for solving (2.1), the method of successive approximations is the most popular one (see [1]). It is based on the idea that the set of successive functions defined by

$$y_{n+1}(t) = \phi(t) - h\int_0^t k(t-\tau) F(y_n(\tau))\ d\tau\ , \qquad (2.2)$$

where $k(t-\tau)$ is equal to the term in braces in (2.1), converges to a solution of (2.1) in every finite interval of time. In the following section, the solution method will be outlined, and at the end of the chapter, general comments will be made on the technique.

B. OUTLINE OF THE METHOD

Consider the time domain in which integral equation (2.1) is to be solved. Suppose the domain is partitioned into N intervals. For the first time interval, $0 \le t \le t_1$, the approximate solution of the integral equation can be obtained by using the iteration procedure

$$U_{n+1}(1,t) = \phi(t) - h\int_0^t k(t-\tau)F(U_n(1,\tau))\, d\tau \qquad (2.3)$$

until the error between two approximations is less than a predefined small number. Next, consider the second time interval, $t_1 \le t \le t_2$. In this interval,

$$U_{n+1}(1,t) = \phi(t) - h\int_0^t k(t-\tau)F(U_n(1,\tau))\, d\tau \qquad (2.4)$$

can be broken into

$$U_{n+1}(1,t) = \phi(t) - h\int_0^{t_1} k(t-\tau)F(U_n(1,\tau))\, d\tau$$

$$- h\int_{t_1}^t k(t-\tau)F(U_n(1,\tau))\, d\tau\ . \qquad (2.5)$$

Since U(1,t) is determined for $0 \le t \le t_1$, the first integral,

$$h\int_0^t k(t-\tau)F(U_n(1,\tau))\, d\tau\ ,$$

is known(approximately), and thus the iteration procedure is only needed for the second integral. In gener-l, for the i-*th* time interval, the iteration procedure is given by

$$U_{n+1}(1,t) = \phi(t) - h\int_0^{t_{i-1}} k(t-\tau)F(U_n(1,\tau))\ d\tau$$

$$- h\int_{t_{i-1}}^{t} k(t-\tau)F(U_n(1,\tau))\ d\tau\ , \qquad (2.6)$$

where $t_{i-1} \leq t \leq t_i$.

As the procedure continues, the surface temperature is found for all desired times. One may notice that as the algorithm is carried out, the singularity of the Volterra part of the integral equation creates difficulty. Appropriately, one has to know the nature of the singularity which the kernel possesses. To illustrate the idea, consider the integral

$$\int_a^b \frac{f(z)}{\sqrt{b-z}}\ dz\ . \qquad (2.7)$$

This integral is often found in the integral representation of the stated problem. Integral (2.7) possesses a singularity which can be removed by the use of the transformation

$$z = a + (b-a)(1-\chi^2)\ . \qquad (2.8)$$

Then, by using a suitable Gaussian quadrature formula, the integral can be evaluated accurately. Normally, one usually comes up with an integral with a stronger singularity.

The iteration procedure outlined above needs a starting value. Generally, the algorithm will converge faster to the exact solution if the starting value is close to the exact solution. Thus, the choice of initial approximation is crucial for convergence. Based on the fact that the solution of the stated problem is continuous, one can choose the temperature at the previous time level as the first approximation of the method when a small time step is used.

The method of successive approximations has been applied (in Chapter II) to solve integral equation (2.1). In particular, a method used to tackle a simple type of singularity, which one may encounter when evaluating the Volterra part numerically, has been discussed. Since numerical integration is one of the key steps in the method, the choice of the numerical integration scheme does affect the overall performance of the algorithm. One can improve the accuracy of the successive approximations method by appropriately choosing a numerical quadrature that can best deal with the singularity found in the integral equation. Even though the procedure outlined above may seem simple, it has been shown that the method is impractical for large times [3].

III. THE RUNGE-KUTTA METHOD

A. INTRODUCTION

This chapter considers another way to deal with the integral equation for the stated heat conduction problem, namely the Runge-Kutta method which was first introduced by Crosbie and Viskanta [5] in 1968. The basic idea of the method is based on an approximation of the kernel by a separable kernel. The integral equation is differentiated with respect to time and transformed into a nonlinear differential equation. The Runge-Kutta method is a well known numerical scheme for solutions of ordinary differential equations. In order to employ the method the surface temperature at a desired time must be determined. The order of approximation of the method is determined by the order of the ordinary differential equation. What differentiates this method from the other numerical schemes is instead of solving an integral equation directly, the Volterra integral equation is first reduced to a system of nonlinear ordinary differential equations and then solved numerically. The method is not exact since the approximation of the kernel is not practical if time steps are small. The accuracy of the approximation of the kernel increases with time and order. In the next section, the method will be outlined in detail as it is applied to the integral equation (1.72). In addition, as an example, the formulas for the third and the fifth order versions of the method will be presented explicitly.

B. OUTLINE OF THE METHOD

Consider the integral equations derived in Chapter I. Generally, the integral representation for the dimensionless surface temperature, U(1,t), of the body that we have considered can be written as

$$U(1,t) = \phi(t) - \int_0^t k(t-\tau)F(U(1,\tau))\, d\tau , \tag{3.1}$$

where

$$k(t-\tau) = \beta_0 + \sum_{k=1}^{\infty} \beta_k e^{-\alpha_k^2(t-\tau)} . \tag{3.2}$$

The function F(U(1,t)) is the surface heat flux; the α_k's and β_k's are eigenvalues and coefficients, respectively. As shown in chapter 1, the infinite series k(t-τ) has the property

$$\lim_{t\to\infty} k(t) = \beta_0 . \tag{3.3}$$

This is a necessary condition for an integral equation to which the method is applied. Now, assume $\phi(t)$ is a bounded differentiable function. The $N^{\underline{th}}$-order approximation of $k(t-\tau)$ is given by taking the first N terms of the infinite sum. So, (3.2) becomes

$$k(t-\tau) \approx \beta_0 + \sum_{k=1}^{N} \beta_k e^{-\alpha_k^2(t-\tau)} \ . \tag{3.4}$$

Substitute (3.4) in (3.1) and let

$$I_k(t) = e^{-\alpha_k^2 t}\int_0^t e^{\alpha_k^2 \tau} F(U(1,\tau))\ d\tau \ . \tag{3.5}$$

U(1,t) becomes

$$U(1,t) = \phi(t) - \beta_0 \int_0^t F(U(1,\tau))\ d\tau - \sum_{k=1}^{N} \beta_k I_k(t) \ . \tag{3.6}$$

Differentiating with respect to time, equation (3.6) becomes

$$U^{(1)}(1,t) = \phi^{(1)}(t) - \beta_0 F(U(1,t))$$
$$- \sum_{k=1}^{N} \beta_k [F(U(1,t)) - \alpha_k^2 I_k(t)] \ . \tag{3.7}$$

$$U^{(2)}(1,t) = \phi^{(2)}(t) - \beta_0 F^{(1)}(U(1,t))$$
$$- \sum_{k=1}^{N} \beta_k [F^{(1)}(U(1,t)) - \alpha_k^2 F(U(1,t)) + \alpha_k^4 I_k(t)] \ , \tag{3.7a}$$

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$$U^{(m)}(1,t) = \phi^{(m)}(t) - \beta_0 F^{(m-1)}(U(1,t))$$

$$- \sum_{k=1}^{N} \beta_k [\sum_{i=0}^{m-1} (-1)^i \alpha_k^{2i} F^{(m-1-i)}(U(1,t)) + (-1)^m \alpha_k^{2m} I_k(t)] \ . \quad (3.8)$$

In general, the $N^{\underline{th}}$-order approximation of the surface temperature U(1,t) is determined by assuming that k(t-τ) in (3.1) takes the form of (3.4), in which only the first N terms of the infinite sum are considered, so that U(1,t) is approximated by (3.6). Then, by performing N+1 differentiations of (3.6), the resulting system of integrodifferential equations obtained by substitution of 1, ..., N+1 for m in (3.8) is found to be

$$U^{(1)}(1,t) = \phi^{(1)}(t) - \beta_0 F(U(1,t))$$

$$- \sum_{k=1}^{N} \beta_k [F(U(1,t)) - \alpha_k^2 I_k(t)] \ , \quad (3.9)$$

$$U^{(2)}(1,t) = \phi^{(2)}(t) - \beta_0 F^{(1)}(U(1,t))$$

$$- \sum_{k=1}^{N} \beta_k [F^{(1)}(U(1,t)) - \alpha_k^2 F(U(1,t)) + \alpha_k^4 I_k(t)] \ , \quad (3.10)$$

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$$U^{(N)}(1,t) = \phi^{(N)}(t) - \beta_0 F^{(N-1)}(U(1,t))$$

$$- \sum_{k=1}^{N} \beta_k [\sum_{i=0}^{N-1} (-1)^i \alpha_k^{2i} F^{(N-1-i)}(U(1,t))$$

$$+ (-1)^N \alpha_k^{2N} I_k(t)] \ , \qquad (3.11)$$

$$U^{(N+1)}(1,t) = \phi^{(N+1)}(t) - \beta_0 F^{(N)}(U(1,t))$$

$$- \sum_{k=1}^{N} \beta_k [\sum_{i=0}^{N} (-1)^i \alpha_k^{2i} F^{(N-i)}(U(1,t))$$

$$+ (-1)^{N+1} \alpha_k^{2N+2} I_k(t)] \ . \qquad (3.12)$$

To eliminate integrals $I_k(t)$, where $k=1,\ldots,N$, from (3.12), consider the first N derivatives of the surface temperature which are given by (3.9)-(3.11). Rearrangement of the terms in (3.9)-(3.11) yields

$$\sum_{k=1}^{N} \alpha_k^2 \beta_k I_k(t) = U^{(1)}(1,t) - \phi^{(1)}(t) + \beta_0 F(U(1,t))$$

$$+ \sum_{k=1}^{N} \beta_k F(U(1,t)) \ , \qquad (3.13)$$

$$- \sum_{k=1}^{N} \alpha_k^4 \beta_k I_k(t) = U^{(2)}(1,t) - \phi^{(2)}(t) + \beta_0 F^{(1)}(U(1,t))$$

$$+ \sum_{k=1}^{N} \beta_k \left[F^{(1)}(U(1,t)) - \alpha_k^2 F(U(1,t))\right] \ , \qquad (3.14)$$

$$\vdots$$

$$(-1)^N \sum_{k=1}^{N} \alpha_k^{2N-2}\beta_k I_k(t) = U^{(N-1)}(1,t) - \phi^{(N-1)}(t) + \beta_0 F^{(N-2)}(U(1,t))$$
$$+ \sum_{k=1}^{N} \beta_k \Big[\sum_{i=0}^{N-2} (-1)^i \alpha_k^{(2i)} F^{(N-2-i)}(U(1,t))\Big] , \qquad (3.15)$$

$$(-1)^{N+1} \sum_{k=1}^{N} \alpha_k^{2N}\beta_k I_k(t) = U^{(N)}(1,t) - \phi^{(N)}(t) + \beta_0 F^{(N-1)}(U(1,t))$$
$$+ \sum_{k=1}^{N} \beta_k \Big[\sum_{i=0}^{N-1} (-1)^i \alpha_k^{(2i)} F^{(N-1-i)}(U(1,t))\Big] . \qquad (3.16)$$

In matrix representation we obtain,

$$\begin{bmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \dots & \beta_N\alpha_N^2 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & \dots & -\beta_N\alpha_N^4 \\ \vdots & \vdots & \dots & \vdots \\ (-1)^N\beta_1\alpha_1^{2N-2} & (-1)^N\beta_2\alpha_2^{2N-2} & \dots & (-1)^N\beta_N\alpha_N^{2N-2} \\ (-1)^{N+1}\beta_1\alpha_1^{2N} & (-1)^{N+1}\beta_2\alpha_2^{2N} & \dots & (-1)^{N+1}\beta_N\alpha_N^{2N} \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \\ \vdots \\ I_{N-1}(t) \\ I_N(t) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{N-1} \\ B_N \end{bmatrix} , \qquad (3.17)$$

where $B_1, \dots, B_N$ are defined as

$$B_1 = U^{(1)}(1,t) - \phi^{(1)}(t) + \beta_0 F(U(1,t))$$
$$+ \sum_{k=1}^{N} \beta_k F(U(1,t)) , \qquad (3.18)$$

$$B_2 = U^{(2)}(1,t) - \phi^{(2)}(t) + \beta_0 F^{(1)}(U(1,t))$$

$$+ \sum_{k=1}^{N} \beta_k [F^{(1)}(U(1,t)) - \alpha_k^2 F(U(1,t))] \ , \qquad (3.19)$$

$$\vdots$$

$$B_{N-1} = U^{(N-1)}(1,t) - \phi^{(N-1)}(t) + \beta_0 F^{(N-2)}(U(1,t))$$

$$+ \sum_{k=1}^{N} \beta_k \Big[\sum_{i=0}^{N-2} (-1)^i \alpha_k^{2i} F^{(N-2-i)}(U(1,t))\Big] \ , \qquad (3.20)$$

$$B_N = U^{(N)}(1,t) - \phi^{(N)}(t) + \beta_0 F^{(N-1)}(U(1,t))$$

$$+ \sum_{k=1}^{N} \beta_k \Big[\sum_{i=0}^{N-1} (-1)^i \alpha_k^{2i} F^{(N-1-i)}(U(1,t))\Big] \ . \qquad (3.21)$$

Now, let

$$A = \begin{bmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \ldots & \beta_N\alpha_N^2 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & \ldots & -\beta_N\alpha_N^4 \\ \vdots & \vdots & \ldots & \vdots \\ (-1)^N\beta_1\alpha_1^{2N-2} & (-1)^N\beta_2\alpha_2^{2N-2} & \ldots & (-1)^N\beta_N\alpha_N^{2N-2} \\ (-1)^{N+1}\beta_1\alpha_1^{2N} & (-1)^{N+1}\beta_2\alpha_2^{2N} & \ldots & (-1)^{N+1}\beta_N\alpha_N^{2N} \end{bmatrix} . \qquad (3.22)$$

By Cramer's rule, $I_1(t), \ldots, I_N(t)$ can be expressed as a quotient of $N \times N$ determinants, given by

$$I_1(t) = \frac{\begin{vmatrix} B_1 & \beta_2\alpha_2^2 & \cdots & \beta_N\alpha_N^2 \\ B_2 & -\beta_2\alpha_2^4 & \cdots & -\beta_N\alpha_N^4 \\ \vdots & \vdots & \cdots & \vdots \\ B_{N-1} & (-1)^N\beta_2\alpha_2^{2N-2} & \cdots & (-1)^N\beta_N\alpha_N^{2N-2} \\ B_N & (-1)^{N+1}\beta_2\alpha_2^{2N} & \cdots & (-1)^{N+1}\beta_N\alpha_N^{2N} \end{vmatrix}}{Det(A)}, \tag{3.23}$$

$$I_2(t) = \frac{\begin{vmatrix} \beta_1\alpha_1^2 & B_1 & \beta_3^2\alpha_3^2 & \cdots & \beta_N\alpha_N^2 \\ -\beta_1\alpha_1^4 & B_2 & -\beta_3\alpha_3^4 & \cdots & -\beta_N\alpha_N^4 \\ \vdots & \vdots & \cdots & \vdots & \\ (-1)^N\beta_1\alpha_1^{2N-2} & B_{N-1} & (-1)^N\beta_3\alpha_3^{2N-2} & \cdots & (-1)^N\beta_N\alpha_N^{2N-2} \\ (-1)^{N+1}\beta_1\alpha_1^{2N} & B_N & (-1)^{N+1}\beta_3\alpha_3^{2N} & \cdots & (-1)^{N+1}\beta_N\alpha_N^{2N} \end{vmatrix}}{Det(A)}, \tag{3.24}$$

$$\vdots$$

$$I_{N-1}(t) = \frac{\begin{vmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \cdots & B_1 & \beta_N\alpha_N^2 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & \cdots & B_2 & -\beta_N\alpha_N^4 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (-1)^N\beta_1\alpha_1^{2N-2} & (-1)^N\beta_2\alpha_2^{2N-2} & \cdots & B_{N-1} & (-1)^N\beta_N\alpha_N^{2N-2} \\ (-1)^{N+1}\beta_1\alpha_1^{2N} & (-1)^{N+1}\beta_2\alpha_2^{2N} & \cdots & B_N & (-1)^{N+1}\beta_N\alpha_N^{2N} \end{vmatrix}}{Det(A)}, \tag{3.25}$$

$$I_N(t) = \frac{\begin{vmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \cdots & \beta_{N-1}\alpha_{N-1}^2 & B_1 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & \cdots & -\beta_{N-1}\alpha_{N-1}^4 & B_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (-1)^N\beta_1\alpha_1^{2N-2} & (-1)^N\beta_2\alpha_2^{2N-2} & \cdots & (-1)^N\beta_{N-1}\alpha_{N-1}^{2N-2} & B_{N-1} \\ (-1)^{N+1}\beta_1\alpha_1^{2N} & (-1)^{N+1}\beta_2\alpha_2^{2N} & \cdots & (-1)^{N+1}\beta_{N-1}\alpha_{N-1}^{2N} & B_N \end{vmatrix}}{Det(A)}, \tag{3.26}$$

where

$$Det\ (A) = \begin{vmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \cdots & \beta_N\alpha_N^2 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & \cdots & -\beta_N\alpha_N^4 \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^N\beta_1\alpha_1^{2N-2} & (-1)^N\beta_2\alpha_2^{2N-2} & \cdots & (-1)^N\beta_N\alpha_N^{2N-2} \\ (-1)^{N+1}\beta_1\alpha_1^{2N} & (-1)^{N+1}\beta_2\alpha_2^{2N} & \cdots & (-1)^{N+1}\beta_N\alpha_N^{2N} \end{vmatrix} . \qquad (3.27)$$

Next, by using only the fundamental properties of determinants, (3.23)-(3.26) can be simplified as a quotient of (N-1)×(N-1) determinants (see appendix-A for further detail). Define

$$Det(A') =$$

$$\begin{vmatrix} -(\alpha_2^2-\alpha_1^2) & -(\alpha_3^2-\alpha_1^2) & \cdots & -(\alpha_N^2-\alpha_1^2) \\ (\alpha_2^4-\alpha_1^4) & (\alpha_3^4-\alpha_1^4) & \cdots & (\alpha_N^2-\alpha_1^4) \\ \vdots & \vdots & \vdots & \\ (-1)^N(\alpha_2^{2N-4}-\alpha_1^{2N-4}) & (-1)^N(\alpha_3^{2N-4}-\alpha_1^{2N-4}) & \cdots & (-1)^N(\alpha_N^{2N-4}-\alpha_1^{2N-4}) \\ (-1)^{N+1}(\alpha_2^{2N-2}-\alpha_1^{2N-2}) & (-1)^{N+1}(\alpha_3^{2N-2}-\alpha_1^{2N-2}) & \cdots & (-1)^{N+1}(\alpha_N^{2N-2}-\alpha_1^{2N-2}) \end{vmatrix} .$$

(3.28)

The formulas are found to be

$$I_1(t) = \frac{\begin{vmatrix} (B_2+\alpha_2^2 B_1) & -(\alpha_3^2-\alpha_2^2) & \cdots & -(\alpha_N^2-\alpha_2^2) \\ (B_3-\alpha_2^4 B_1) & (\alpha_3^4-\alpha_2^4) & \cdots & (\alpha_N^4-\alpha_2^4) \\ \vdots & \vdots & \cdots & \vdots \\ (B_{N-1}-(-1)^N \alpha_2^{2N-4} B_1) & (-1)^N(\alpha_3^{2N-4}-\alpha_2^{2N-4}) & \cdots & (-1)^N(\alpha_N^{2N-4}-\alpha_2^{2N-4}) \\ (B_N-(-1)^{N+1}\alpha_2^{2N-2} B_1) & (-1)^{N+1}(\alpha_3^{2N-2}-\alpha_2^{2N-2}) & \cdots & (-1)^{N+1}(\alpha_N^{2N-2}-\alpha_2^{2N-2}) \end{vmatrix}}{\beta_1 \alpha_1^2 \, Det(A')}, \tag{3.29}$$

and for k = 2, ..., N-1,

(continued on next page)

$$
I_k(t) = \frac{(-1)^k \begin{vmatrix}
-(\alpha_1^2-\alpha_{k+1}^2) & \cdots & -(\alpha_{k-1}^2-\alpha_{k+1}^2) & (B_1+\alpha_{k+1}^2 B_1) & -(\alpha_{k+2}^2-\alpha_{k+1}^2) & \cdots & -(\alpha_N^2-\alpha_{k+1}^2) \\
(\alpha_1^4-\alpha_{k+1}^4) & \cdots & (\alpha_{k-1}^4-\alpha_{k+1}^4) & (B_3-\alpha_{k+1}^4 B_1) & (\alpha_{k+2}^4-\alpha_{k+1}^4) & \cdots & (\alpha_N^4-\alpha_{k+1}^4) \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
(-1)^N(\alpha_1^{2N-4}-\alpha_{k+1}^{2N-4}) & \cdots & (-1)^N(\alpha_{k-1}^{2N-4}-\alpha_{k+1}^{2N-4}) & (B_{N-1}-(-1)^N\alpha_{k+1}^{2N-4}B_1) & (-1)^N(\alpha_{k+2}^{2N-4}-\alpha_{k+1}^{2N-4}) & \cdots & (-1)^N(\alpha_N^{2N-4}-\alpha_{k+1}^{2N-4}) \\
(-1)^{N+1}(\alpha_1^{2N-2}-\alpha_{k+1}^{2N-2}) & \cdots & (-1)^{N+1}(\alpha_{k-1}^{2N-2}-\alpha_{k+1}^{2N-2}) & (B_N-(-1)^{N+1}\alpha_{k+1}^{2N-2}B_1) & (-1)^{N+1}(\alpha_{k+2}^{2N-2}-\alpha_{k+1}^{2N-2}) & \cdots & (-1)^{N+1}(\alpha_N^{2N-2}-\alpha_{k+1}^{2N-2})
\end{vmatrix}}{\beta_N \alpha_N^2 \, Det(\Lambda')} \tag{3.30}
$$

and

$$I_N(t) =$$

$$\frac{\begin{vmatrix} -(\alpha_1^2-\alpha_{N-1}^2) & \dots & -(\alpha_{N-2}^2-\alpha_{N-1}^2) & (B_2+\alpha_{N-1}^2 B_1) \\ (\alpha_1^4-\alpha_{N-1}^4) & \dots & (\alpha_{N-2}^4-\alpha_{N-1}^4) & (B_3-\alpha_{N-1}^4 B_1) \\ \vdots & \dots & \vdots & \dots \\ (-1)^N(\alpha_1^{2N-4}-\alpha_{N-1}^{2N-4}) & \dots & (-1)^N(\alpha_{N-2}^{2N-4}-\alpha_{N-1}^{2N-4}) & (B_{N-1}-(-1)^N\alpha_{N-1}^{2N-4}B_1) \\ (-1)^{N+1}(\alpha_1^{2N-2}-\alpha_{N-1}^{2N-2}) & \dots & (-1)^{N+1}(\alpha_{N-2}^{2N-2}-\alpha_{N-1}^{2N-2}) & (B_N-(-1)^{N+1}\alpha_{N-1}^{2N-2}B_1) \end{vmatrix}}{\beta_N\alpha_N^2\, Det(A')} \ . \tag{3.31}$$

Thus, $I_k(t)$, where k = 1, ..., N, in equation (3.12) can be determined explicitly by formulas (3.29) through (3.31). Hence, the integrodifferential equation (3.12) is reduced to a $(N+1)^{\underline{th}}$ order nonlinear ordinary differential equation

$$\begin{aligned} U^{(N+1)}(1,t) &= \phi^{(N+1)}(t) - \beta_0 F^{(N)}(U(1,t)) \\ &\quad - \textstyle\sum_{k=1}^{N} \beta_k [\sum_{i=0}^{N} (-1)^i \alpha_k^{2i} F^{(N-i)}(U(1,t)) \\ &\quad + (-1)^{N+1}\alpha_k^{2N+2} I_k(t)] \ , \end{aligned} \tag{3.32}$$

with the initial values

$$U(1,0) = \phi(0) \ , \tag{3.33}$$

$$\begin{aligned} U^{(1)}(1,0) &= \phi^{(1)}(0) - \beta_0 F(U(1,0)) \\ &\quad - \textstyle\sum_{k=1}^{N} \beta_k [F(U(1,0))] \ , \end{aligned} \tag{3.34}$$

$$U^{(2)}(1,0) = \phi^{(2)}(0) - \beta_0 F^{(1)}(U(1,0))$$

$$- \sum_{k=1}^{N} \beta_k [F^{(1)}(U(1,0)) - \alpha_k^2 F(U(1,0))] \;, \qquad (3.35)$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$U^{(N)}(1,0) = \phi^{(N)}(0) - \beta_0 F^{(N-1)}(U(1,0))$$

$$- \sum_{k=1}^{N} \beta_k [\sum_{i=0}^{N-1} (-1)^i \alpha_k^{2i} F^{(N-1-i)}(U(1,0))] \;, \qquad (3.36)$$

which can be obtained by putting t = 0 in (3.9) through (3.12), and $I_1(t)$, ..., $I_N(t)$ are determined by formulas (3.29) through (3.31).

The Runge-Kutta method is then applied to the nonlinear ordinary differential equation (3.32) with initial conditions (3.33) through (3.36). Hence, the N^{th} order approximation of the surface temperature will be given by the numerical solution of a $(N+1)^{th}$ order nonlinear ordinary differential equation.

As an example, in the next section, formulas for the third and the fifth order approximation of the method will be presented. Details of the derivation of the equations will not be produced, and only major results will be given.

C. THE THIRD ORDER APPROXIMATION

The third order approximation of the surface temperature of (3.1) is given by the numerical solution of the following fourth order nonlinear ordinary differential equation

$$\begin{aligned} U^{(4)}(1,t) = \phi^{(4)}(t) &- (\beta_0+\beta_1+\beta_2+\beta_3)F^{(3)}(U(1,t)) \\ &+ (\beta_1\alpha_1^2+\beta_2\alpha_2^2+\beta_3\alpha_3^2)F^{(2)}(U(1,t)) \\ &- (\beta_1\alpha_1^4+\beta_2\alpha_2^4+\beta_3\alpha_3^4)F^{(1)}(U(1,t)) \\ &+ (\beta_1\alpha_1^6+\beta_2\alpha_2^6+\beta_3\alpha_3^6)F(U(1,t)) \\ &- (\beta_1\alpha_1^8 I_1(t)+\beta_2\alpha_2^8 I_2(t)+\beta_3\alpha_3^8 I_3(t)) \end{aligned} \tag{3.37}$$

with initial conditions

$$U(1,0) = \phi(0)\ , \tag{3.38a}$$

$$U^{(1)}(1,0) = \phi^{(1)}(0) - (\beta_0+\beta_1+\beta_2+\beta_3)F(\phi(0))\ , \tag{3.38b}$$

$$\begin{aligned} U^{(2)}(1,0) = \phi^{(2)}(0) &- (\beta_0+\beta_1+\beta_2+\beta_3)F^{(1)}(\phi(0)) \\ &+ (\beta_1\alpha_1^2+\beta_2\alpha_2^2+\beta_3\alpha_3^2)F(\phi(0))\ , \end{aligned} \tag{3.39}$$

and

$$U^{(3)}(1,0) = \phi^{(3)}(0) - (\beta_0 + \beta_1 + \beta_2 + \beta_3)F^{(2)}(\phi(0))$$
$$+ (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2)F^{(1)}(\phi(0))$$
$$- (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4)F(\phi(0)) \ . \tag{3.40}$$

$I_1(t)$, $I_2(t)$, and $I_3(t)$ in (3.37) are

$$I_1(t) = -\frac{\begin{vmatrix} (B_2+\alpha_2^2B_1) & -(\alpha_3^2-\alpha_2^2) \\ (B_3-\alpha_2^4B_1) & (\alpha_3^4-\alpha_2^4) \end{vmatrix}}{\beta_1\alpha_1^2 \ Det(A')} \ , \tag{3.41}$$

$$I_2(t) = \frac{\begin{vmatrix} (\alpha_1^2-\alpha_3^2) & (B_2+\alpha_3^2B_1) \\ (\alpha_1^4-\alpha_3^4) & (B_3-\alpha_3^4B_1) \end{vmatrix}}{\beta_2\alpha_2^2 \ Det(A')} \ , \tag{3.42}$$

$$I_3(t) = -\frac{\begin{vmatrix} (\alpha_1^2-\alpha_2^2) & (B_2+\alpha_2^2B_1) \\ (\alpha_1^4-\alpha_2^4) & (B_3-\alpha_2^4B_1) \end{vmatrix}}{\beta_3\alpha_3^2 \ Det(A')} \ , \tag{3.43}$$

where

$$B_1 = U^{(1)}(1,t) - \phi^{(1)}(t) + (\beta_0 + \beta_1 + \beta_2 + \beta_3)F(U(1,t)) \ ,$$

$$B_2 = U^{(2)}(1,t) - \phi^{(2)}(t) - (\beta_0 + \beta_1 + \beta_2 + \beta_3) F^{(1)}(U(1,t))$$
$$+ (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2) F(U(1,t)) ,$$

$$B_3 = U^{(3)}(1,t) - \phi^{(3)}(t) + (\beta_0 + \beta_1 + \beta_2 + \beta_3) F^{(2)}(U(1,t))$$
$$- (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2) F^{(1)}(U(1,t))$$
$$+ (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4) F(U(1,t)) ,$$

and

$$Det(A') = \begin{vmatrix} -(\alpha_2^2-\alpha_1^2) & -(\alpha_3^2-\alpha_1^2) \\ (\alpha_2^4-\alpha_1^4) & (\alpha_3^4-\alpha_1^4) \end{vmatrix} .$$

D. THE FIFTH ORDER APPROXIMATION

The fifth order approximation of the surface temperature of (3.1) is given by the numerical solution of the following sixth order nonlinear differential equation

$$U^{(6)}(1,t) = \phi^{(6)}(t) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5)F^{(5)}(U(1,t))$$

$$+ (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2)F^{(4)}(U(1,t))$$

$$- (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4 + \beta_4\alpha_4^4 + \beta_5\alpha_5^4)F^{(3)}(U(1,t))$$

$$+ (\beta_1\alpha_1^6 + \beta_2\alpha_2^6 + \beta_3\alpha_3^6 + \beta_4\alpha_4^6 + \beta_5\alpha_5^6)F^{(2)}(U(1,t))$$

$$- (\beta_1\alpha_1^8 + \beta_2\alpha_2^8 + \beta_3\alpha_3^8 + \beta_4\alpha_4^8 + \beta_5\alpha_5^8)F^{(1)}(U(1,t))$$

$$+ (\beta_1\alpha_1^{10} + \beta_2\alpha_2^{10} + \beta_3\alpha_3^{10} + \beta_4\alpha_4^{10} + \beta_5\alpha_5^{10})F(U(1,t))$$

$$- (\beta_1\alpha_1^{12}I_1(t) + \beta_2\alpha_2^{12}I_2(t) + \beta_3\alpha_3^{12}I_3(t)$$

$$+ \beta_4\alpha_4^{12}I_4(t) + \beta_5\alpha_5^{12}I_5(t)) \,, \qquad (3.44)$$

with initial conditions

$$U(1,0) = \phi(0) \,,$$

$$U^{(1)}(1,0) = \phi^{(1)}(0) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5)F(\phi(0)) \,,$$

$$U^{(2)}(1,0) = \phi^{(2)}(0) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5)F^{(1)}(\phi(0))$$

$$+ (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2)F(\phi(0)) \,,$$

$$U^{(3)}(1,0) = \phi^{(3)}(t) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F^{(2)}(\phi(0))$$

$$+ (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2) F^{(1)}(\phi(0))$$

$$- (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4 + \beta_4\alpha_4^4 + \beta_5\alpha_5^4) F(\phi(0)) \ ,$$

$$U^{(4)}(1,0) = \phi^{(4)}(0) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F^{(3)}(\phi(0))$$

$$+ (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2) F^{(2)}(\phi(0))$$

$$- (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4 + \beta_4\alpha_4^4 + \beta_5\alpha_5^4) F^{(1)}(\phi(0))$$

$$+ (\beta_1\alpha_1^6 + \beta_2\alpha_2^6 + \beta_3\alpha_3^6 + \beta_4\alpha_4^6 + \beta_5\alpha_5^6) F(\phi(0)) \ ,$$

$$U^{(5)}(1,0) = \phi^{(5)}(0) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F^{(4)}(\phi(0))$$

$$+ (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2) F^{(3)}(\phi(0))$$

$$- (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4 + \beta_4\alpha_4^4 + \beta_5\alpha_5^4) F^{(2)}(\phi(0))$$

$$+ (\beta_1\alpha_1^6 + \beta_2\alpha_2^6 + \beta_3\alpha_3^6 + \beta_4\alpha_4^6 + \beta_5\alpha_5^6) F^{(1)}(\phi(0))$$

$$- (\beta_1\alpha_1^8 + \beta_2\alpha_2^8 + \beta_3\alpha_3^8 + \beta_4\alpha_4^8 + \beta_5\alpha_5^8) F(\phi(0)) \ .$$

$I_1(t)$, $I_2(t)$, $I_3(t)$, $I_4(t)$, and $I_5(t)$ in 3.44 are

$$I_1(t) = -\frac{\begin{vmatrix} (B_2+\alpha_2^2 B_1) & -(\alpha_3^2-\alpha_2^2) & -(\alpha_4^2-\alpha_2^2) & -(\alpha_5^2-\alpha_2^2) \\ (B_3-\alpha_2^4 B_1) & (\alpha_3^4-\alpha_2^4) & (\alpha_4^4-\alpha_2^4) & (\alpha_5^4-\alpha_2^4) \\ (B_4+\alpha_2^6 B_1) & -(\alpha_3^6-\alpha_2^6) & -(\alpha_4^6-\alpha_2^6) & -(\alpha_5^6-\alpha_2^6) \\ (B_5-\alpha_2^8 B_1) & (\alpha_3^8-\alpha_2^8) & (\alpha_4^8-\alpha_2^8) & (\alpha_5^8-\alpha_2^8) \end{vmatrix}}{\beta_1 \alpha_1^2 \, Det(A')}, \qquad (3.45)$$

$$I_2(t) = -\frac{\begin{vmatrix} -(\alpha_1^2-\alpha_3^2) & (B_2+\alpha_3^2 B_1) & -(\alpha_4^2-\alpha_3^2) & -(\alpha_5^2-\alpha_3^2) \\ (\alpha_1^4-\alpha_3^4) & (B_3-\alpha_3^4 B_1) & (\alpha_4^4-\alpha_3^4) & (\alpha_5^4-\alpha_3^4) \\ -(\alpha_1^6-\alpha_3^6) & (B_4+\alpha_3^6 B_1) & -(\alpha_4^6-\alpha_3^6) & -(\alpha_5^6-\alpha_3^6) \\ (\alpha_1^8-\alpha_3^8) & (B_5-\alpha_3^8 B_1) & (\alpha_4^8-\alpha_3^8) & (\alpha_5^8-\alpha_3^8) \end{vmatrix}}{\beta_2 \alpha_2^2 \, Det(A')}, \qquad (3.46)$$

$$I_3(t) = \frac{\begin{vmatrix} -(\alpha_1^2-\alpha_4^2) & -(\alpha_2^2-\alpha_4^2) & (B_2+\alpha_4^2 B_1) & -(\alpha_5^2-\alpha_4^2) \\ (\alpha_1^4-\alpha_4^4) & (\alpha_2^4-\alpha_4^4) & (B_3-\alpha_4^4 B_1) & (\alpha_5^4-\alpha_4^4) \\ -(\alpha_1^6-\alpha_4^6) & -(\alpha_2^6-\alpha_4^6) & (B_4+\alpha_4^6 B_1) & -(\alpha_5^6-\alpha_4^6) \\ (\alpha_1^8-\alpha_4^8) & (\alpha_2^8-\alpha_4^8) & (B_5-\alpha_4^8 B_1) & (\alpha_5^8-\alpha_4^8) \end{vmatrix}}{\beta_3 \alpha_3^2 \, Det(A')}, \qquad (3.47)$$

$$I_4(t) = \frac{\begin{vmatrix} -(\alpha_1^2-\alpha_5^2) & -(\alpha_2^2-\alpha_5^2) & -(\alpha_3^2-\alpha_5^2) & (B_2+\alpha_5^2 B_1) \\ (\alpha_1^4-\alpha_5^4) & (\alpha_2^4-\alpha_5^4) & (\alpha_3^4-\alpha_5^4) & (B_3-\alpha_5^4 B_1) \\ -(\alpha_1^6-\alpha_5^6) & -(\alpha_2^6-\alpha_5^6) & -(\alpha_3^6-\alpha_5^6) & (B_4+\alpha_5^6 B_1) \\ (\alpha_1^8-\alpha_5^8) & (\alpha_2^8-\alpha_5^8) & (\alpha_3^8-\alpha_5^8) & (B_5-\alpha_5^8 B_1) \end{vmatrix}}{\beta_4 \alpha_4^2 \, Det(A')}, \qquad (3.48)$$

$$I_5(t) = \frac{\begin{vmatrix} -(\alpha_1^2-\alpha_4^2) & -(\alpha_2^2-\alpha_4^2) & -(\alpha_3^2-\alpha_4^2) & (B_2+\alpha_4^2 B_1) \\ (\alpha_1^4-\alpha_4^4) & (\alpha_2^4-\alpha_4^4) & (\alpha_3^4-\alpha_4^4) & (B_3-\alpha_4^4 B_1) \\ -(\alpha_1^6-\alpha_4^6) & -(\alpha_2^6-\alpha_4^6) & -(\alpha_3^6-\alpha_4^6) & (B_4+\alpha_4^6 B_1) \\ (\alpha_1^8-\alpha_4^8) & (\alpha_2^8-\alpha_4^8) & (\alpha_3^8-\alpha_4^8) & (B_5-\alpha_4^8 B_1) \end{vmatrix}}{\beta_5 \alpha_5^2 \, Det(A')} , \qquad (3.49)$$

where

$$Det(A') = \begin{vmatrix} -(\alpha_2^2-\alpha_1^2) & -(\alpha_3^2-\alpha_1^2) & -(\alpha_4^2-\alpha_1^2) & -(\alpha_5^2-\alpha_1^2) \\ (\alpha_2^4-\alpha_1^4) & (\alpha_3^4-\alpha_1^4) & (\alpha_4^4-\alpha_1^4) & (\alpha_5^4-\alpha_1^4) \\ -(\alpha_2^6-\alpha_1^6) & -(\alpha_3^6-\alpha_1^6) & -(\alpha_4^6-\alpha_1^6) & -(\alpha_5^6-\alpha_1^6) \\ (\alpha_2^8-\alpha_1^8) & (\alpha_3^8-\alpha_1^8) & (\alpha_4^8-\alpha_1^8) & (\alpha_5^8-\alpha_1^8) \end{vmatrix} ,$$

$$B_1 = U^{(1)}(1,t) - \phi^{(1)}(t) + (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F(U(1,t))$$

$$B_2 = U^{(2)}(1,t) - \phi^{(2)}(t) + (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F^{(1)}(U(1,t)$$
$$- (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2) F(U(1,t)) ,$$

$$B_3 = U^{(3)}(1,t) - \phi^{(3)}(t) + (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F^{(2)}(U(1,t)$$
$$- (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2) F^{(1)}(U(1,t))$$
$$+ (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4 + \beta_4\alpha_4^4 + \beta_5\alpha_5^4) F(U(1,t)) ,$$

$$B_4 = U^{(4)}(1,t) - \phi^{(4)}(t) + (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F^{(3)}(U(1,t)$$
$$- (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2) F^{(2)}(U(1,t))$$
$$+ (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4 + \beta_4\alpha_4^4 + \beta_5\alpha_5^4) F^{(1)}(U(1,t))$$
$$- (\beta_1\alpha_1^6 + \beta_2\alpha_2^6 + \beta_3\alpha_3^6 + \beta_4\alpha_4^6 + \beta_5\alpha_5^6) F(U(1,t)) \ ,$$

and

$$B_5 = U^{(5)}(1,t) - \phi^{(5)}(t) + (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) F^{(4)}(U(1,t)$$
$$- (\beta_1\alpha_1^2 + \beta_2\alpha_2^2 + \beta_3\alpha_3^2 + \beta_4\alpha_4^2 + \beta_5\alpha_5^2) F^{(3)}(U(1,t))$$
$$+ (\beta_1\alpha_1^4 + \beta_2\alpha_2^4 + \beta_3\alpha_3^4 + \beta_4\alpha_4^4 + \beta_5\alpha_5^4) F^{(2)}(U(1,t))$$
$$- (\beta_1\alpha_1^6 + \beta_2\alpha_2^6 + \beta_3\alpha_3^6 + \beta_4\alpha_4^6 + \beta_5\alpha_5^6) F^{(1)}(U(1,t))$$
$$+ (\beta_1\alpha_1^8 + \beta_2\alpha_2^8 + \beta_3\alpha_3^8 + \beta_4\alpha_4^8 + \beta_5\alpha_5^8) F(U(1,t)) \ .$$

E. REMARKS

In this work the Runge-Kutta method is applied to solve the integral equation resulting from the heat conduction problem with combined convection and radiation. In particular, the nonlinear ordinary differential equation has been determined for both the third and the fifth order approximations. It may be observed that this method is not very practical for calculating the temperature at small time steps. The reason is that the smaller the time one takes, the more terms will be represented correctly, which in turn may result in a high-order nonlinear ordinary differential equation with a very large number of terms. The number of terms could grow to infinity. Thus, the method is usually used to compute the surface temperature at large times where the temperature distribution is in a steady state.

In the following chapter, we will take another approach using a different numerical method, namely the finite difference method. This method is different from the previous numerical techniques in that instead of solving the integral equation, it approximates the partial differential equation and the boundary conditions directly.

IV. THE FINITE DIFFERENCE METHOD

A. INTRODUCTION

The basic idea of the finite difference method is to transform a continuous model into a discrete system by replacing the continuous domain in the model with a denumerable domain. In applying this idea to differential equations, all the derivatives in the equation are simply replaced by finite difference approximations. Thus, the unknowns in the difference equation have a countable domain, and the resulting discrete system is solved numerically.

In the theory of numerical analysis, the significance of the computed solution of a finite difference scheme in relation to approximating the exact solution depends upon three elements. They are consistency, convergence, and stability. Consistency is a condition used to assure that as Δx (the spacing) approaches zero, the truncation error of the scheme also goes to zero. It implies that the finite difference can be an arbitrarily accurate approximation to the derivative. Convergence of the approximation assures that if Δx goes to zero, the difference between the computed and the exact values also goes to zero. In other words, any desired accuracy of the approximated solution can be achieved. The last element is the stability. The stability of a scheme concerns the growth of the errors found in the calculations which are needed to solve the system of linear equations. A scheme is said to be conditionally stable if the roundoff error does not amplify if the time step is under a critical value which is determined by the differential equation considered. In the Lax Equivalence Theorem, the relationships of these three conditions are stated. It says that given a properly posed initial value problem and a finite difference scheme which satisfies the consistency conditions, stability is the necessary and sufficient condition for convergence.

There are many difference approximations and methods for solving discrete systems that are available in numerical analysis. Different choices of approximation and methods of solving the system will lead to differing degrees of accuracy in the approximation of the solution. This chapter will only focus on a particular finite difference scheme used to approximate the governing partial differential equation in the stated problem and an algorithm for solving the discretized system.

B. CRANK-NICHOLSON SCHEME

Suppose a lies between x_0 and x_f and $t \geq t_0$, where x_0 and x_f are some initial and final x-coordinate which brackets the location of concern. Let Δx and Δt be increments of x and t, respectively. The x-t space can be partitioned into a grid network in which the points are given by $x = x_0 + j\Delta x$ and $t =$

$t_0 + n\Delta t$, where $j = 0,1,2,\ldots,N$, with N being the number of nodes, and $n = 0,1,2 \ldots$. When Δx and Δt are constants, the mesh obtained is uniform, and the temperature at $x = x_0 + i\Delta x$, written as x_j, and $t = t_0 + n\Delta x$, written as t_n, is denoted by U_j^n.

As previously mentioned, there are several ways of choosing a finite difference operator for replacing the derivatives. If the average of the forward and backward difference schemes is used for the space discretization and the forward difference scheme is written about the point x_i, $t_{j+½}$, the governing partial differential equation becomes a second order accurate (in both x and t) finite difference equation. It is given by

$$U_{j-1}^{n+1} + (-2 - 2\beta)\,U_j^{n+1} + U_{j+1}^{n+1} = -\,U_{j-1}^{n} + (2 - 2\beta)\,U_j^{n} - U_{j+1}^{n}\ , \qquad (4.1)$$

where

$$\beta = \frac{\Delta x^2}{\Delta t} \quad ,$$

(which is the well known Crank-Nicholson scheme).

Since it is of second order, the truncation error associated with (4.1) is on the order of $o(\Delta x^2 + \Delta t^2)$. Notice that the temperature at time t_{n+1} is a function of unknown and known temperatures at six of the ten points shown on the Fig. 4.1.

$$j = \begin{cases} 0, \ldots, N\text{-}1 \text{ for case } 1 \\ 1, \ldots, N\text{-}1 \text{ for case } 2 \end{cases} . \tag{4.2}$$

To ensure that the oscillation is eliminated, the implicit backward finite difference scheme (which is satisfactory with all types of boundary conditions) is adopted at the boundary, x = 1. The equation at x = 1 is given by

$$U_{N-1}^{n+1} + (-2 - \beta) U_N^{n+1} + U_{N+1}^{n+1} = - \beta U_N^n , \tag{4.3}$$

where β is as before.

There is a fictitious point outside the computational domain in (4.3), that is, the unknown temperature at N+1 is denoted as U_{N+1}^{n+1}. To eliminate that point, use a difference method to approximate the derivative in the radiative boundary condition (1.2c) because

$$\frac{U_{N+1}^{n+1} - U_{N-1}^{n+1}}{2\Delta x} - \alpha_3 U_N^{n+1} = F(U_N^{n+1}) , \tag{4.4}$$

where F is the right hand side of (1.2c). Algebraically manipulating (4.4) yields the following equation

$$U_{N+1}^{n+1} = U_{N-1}^{n+1} + 2\Delta x \alpha_3 U_N^{n+1} + 2\Delta x F(U_N^{n+1}) . \tag{4.5}$$

Substituting (4.5) into (4.3), the resulting expression becomes

$$2U_{N-1}^{n+1} + (-2 - \beta + 2\Delta x\alpha_3)\,U_N^{n+1} = -\,\beta U_N^n - 2\Delta x F(U_N^{n+1}) \quad , \tag{4.6}$$

which is a nonlinear equation in U_N^{n+1}.

Observe that (4.1) and (4.6) constitute a set of simultaneous equations at each time step. In matrix representation, the resulting system is of the form

$$\mathbf{AU} = \mathbf{B} \quad , \tag{4.7}$$

where A is a tridiagonal matrix, B is a vector of all the known values found in each equation, and U is a vector of the unknown temperatures at each space node at a particular moment of time. So, for each time level, the transient temperature is given by the solution of a system of equations.

The Thomas algorithm can be used to solve a tridiagonal system of linear equations. Clearly, all the equations in (4.7) are linear (except the last one). The first half of the algorithm, as given in appendix-B, can be directly applied to the system except for the case where i = N. In the case of i = N, substituting d(N) in the first Do-loop in the expression right after the first loop yields

$$U_N^{n+1} = \frac{d(N) - ratio * d(N\text{-}1)}{b(N)} \quad . \tag{4.8}$$

which implies

$$U_N^{n+1} = \frac{-\beta U_N^n - 2\Delta x F(U_N^{n+1}) - ratio * d(N-1)}{b(N)} , \qquad (4.9)$$

with b(N), d(N-1), and ratio computed in the first Do-loop(reference to appendix-B). Now, (4.9) can be rewritten as

$$\frac{-\beta U_N^n - 2\Delta x F(U_N^{n+1}) - ratio * d(N-1)}{b(N)} - U_N^{n+1} = 0 . \qquad (4.10)$$

Let the left hand side of (4.10) be represented by f. It follows that

$$f(U_N^{n+1}) = 0 \quad . \qquad (4.11)$$

Thus, the update of the surface temperature is the solution of the nonlinear equation (4.11).

In the following section, the cases for a flat plate and a sphere will be considered to obtain the respective tridiagonal systems.

C. TWO SPECIAL CASES

1. The Flat Plate

The parameters corresponding to this case can be found in Chapter I. Applying the finite difference method outlined above to the governing equations leads to the following results.

Consider the Crank-Nicholson scheme for j = 0, ..., N-1. For j = 0,

$$U_{-1}^{n+1} + (-2 - 2\beta)\,U_0^{n+1} + U_1^{n+1} = -\,U_{-1}^{n} + (2-2\beta)\,U_0^{n} - U_1^{n}\,. \qquad (4.12)$$

To eliminate the fictitious points, the boundary condition at x = 0 in discretized form is taken to be

$$\frac{U_1^{n+1} - U_{-1}^{n+1}}{2\Delta x} = 0 \qquad . \qquad (4.13)$$

Thus,

$$U_1^{n+1} = U_{-1}^{n+1} \qquad , \qquad (4.14)$$

and

$$U_1^{n} = U_{-1}^{n} \qquad . \qquad (4.15)$$

Substituting (4.14) and (4.15) in (4.12) produces

$$(-2 - 2\beta)\,U_0^{n+1} + 2U_1^{n+1} = (2-2\beta)\,U_0^{n} - 2U_1^{n}\,. \qquad (4.16)$$

For j = 1, ..., N-1

$$U_0^{n+1} + (-2 - 2\beta) U_1^{n+1} + U_2^{n+1} = - U_0^n + (2-2\beta) U_1^n - U_2^n , \qquad (4.17)$$

$$\vdots$$

$$U_{N-2}^{n+1} + (-2 - 2\beta) U_{N-1}^{n+1} + U_N^{n+1} = - U_{N-2}^n + (2-2\beta) U_{N-1}^n - U_N^n . \qquad (4.18)$$

When j = N, as shown before, the equation becomes

$$2U_{N-1}^{n+1} + (-2 - \beta + 2\Delta x\alpha_3) U_N^{n+1} = - \beta U_N^n - 2\Delta x F(U_N^{n+1}) .$$

In matrix representation, with initial values

$$U_j^0 = 1 , \quad where \ j = 0, \ldots, N ,$$

we have

$$A = \begin{bmatrix} (-2-2\beta) & 2 & 0 & \cdots & \cdots & 0 \\ 1 & (-2-2\beta) & 1 & 0 & \cdots & 0 \\ 0 & 1 & (-2-2\beta) & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & (-2-2\beta) & 1 \\ 0 & 0 & \cdots & 0 & 2 & (-2-\beta+2\Delta x\alpha_3) \end{bmatrix} , \qquad (4.19)$$

$$B = \begin{bmatrix} (2-2\beta)U_0^n - 2U_1^n \\ -U_0^n + (2-2\beta)U_1^n - U_2^n \\ \vdots \\ \vdots \\ -U_{N-2}^n + (2-2\beta)U_{N-1}^n - U_N^n \\ -\beta U_N^n - 2\Delta x F(U_N^{n+1}) \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ \vdots \\ \vdots \\ U_{N-1}^{n+1} \\ U_N^{n+1} \end{bmatrix}. \qquad (4.20)$$

2. The Sphere

The parameters again can be obtained in Chapter I and will not be repeated here. Using the finite difference method outlined above with the governing equations leads to the following:

Consider the Crank-Nicholson scheme for j = 1, ..., N-1.
For j = 1,

$$U_0^{n+1} + (-2 - 2\beta)U_1^{n+1} + U_2^{n+1} = - U_0^n + (2-2\beta)U_1^n - U_2^n . \qquad (4.21)$$

However,

$$U_0^{n+1} = 0 \quad \text{and} \quad U_0^n = 0 \quad . \qquad (4.22)$$

Substituting (4.22) in (4.21) produces

$$(-2 - 2\beta)U_1^{n+1} + U_2^{n+1} = (2-2\beta)U_1^n - U_2^n . \qquad (4.23)$$

For j = 2, ..., N-1

$$U_1^{n+1} + (-2 - 2\beta)U_2^{n+1} + U_3^{n+1} = -U_1^n + (2-2\beta)U_2^n - U_3^n , \qquad (4.24)$$

$$\vdots$$

$$U_{N-2}^{n+1} + (-2 - 2\beta)U_{N-1}^{n+1} + U_N^{n+1} = -U_{N-2}^n + (2-2\beta)U_{N-1}^n - U_N^n . \qquad (4.25)$$

When j = N, as shown before, the backward scheme is used. Thus,

$$2U_{N-1}^{n+1} + (-2 - \beta + 2\Delta x\alpha_3)U_N^{n+1} = -\beta U_N^n - 2\Delta x F(U_N^{n+1}) .$$

In matrix representation, with initial values

$$U_j^0 = j\Delta x , \quad where \ j = 1, \ldots, N ,$$

we have

$$A = \begin{bmatrix} (-2-2\beta) & 1 & 0 & \cdots & \cdots & 0 \\ 1 & (-2-2\beta) & 1 & 0 & \cdots & 0 \\ 0 & 1 & (-2-2\beta) & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & (-2-2\beta) & 1 \\ 0 & 0 & \cdots & 0 & 2 & (-2-\beta+2\Delta x\alpha_3) \end{bmatrix} , \qquad (4.26)$$

$$B = \begin{bmatrix} (2-2\beta)U_1^n - U_2^n \\ -U_1^n + (2-2\beta)U_2^n - U_3^n \\ \vdots \\ \vdots \\ -U_{N-2}^n + (2-2\beta)U_{N-1}^n - U_N^n \\ -\beta U_N^n - 2\Delta x F(U_N^{n+1}) \end{bmatrix}, \quad and \quad U = \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{N-1}^{n+1} \\ U_N^{n+1} \end{bmatrix}. \qquad (4.37)$$

D. STABILITY

Even though the backward analog is implemented on the radiative boundary, according to the numerical experiments, the method still suffers from the problem of oscillations when a large time step is imposed. As far as the author is aware, not a single formula has been developed for the stability criteria of an implicit finite difference scheme with nonlinear boundary conditions. However, two stability formulas of the related problems, which are found in the literature [7], can serve as a guideline in choosing the time step for the problems considered. The first one is due to Lawson and Morris [7]. They deduce the stability criterion for the Crank-Nicholson equation with linear boundary conditions as

$$\Delta t < \frac{2\Delta x}{\pi} \quad . \qquad (4.28)$$

Another stability criterion is due to Milton and Goss [9] who applied the laws of thermodynamics in developing the stability requirement for an explicit finite difference scheme with nonlinear boundary conditions. It turned out that the time step required for the stability is restricted by

$$\Delta t \leq \frac{(\Delta x)^2 \max\{U_N^n\}}{2\{U_{N-1}^n - U_N^n - \Delta x(U_N^n)^4 - \alpha_3 \Delta x U_N^n\}} \ , \qquad (4.29)$$

where the maximum is taken over all n and where U_N^n can be found by setting the following function

$$\frac{\Delta U_N}{\Delta t} = A - BU_N - C(U_N)^4$$

equal to zero, where

$$\Delta U_N = U_N^{n+1} - U_N^n \quad ,$$

$$A = \frac{2U_{N-1}^n}{(\Delta x)^2} \quad ,$$

$$B = \frac{2[\Delta x - 1]}{(\Delta x)^2} \quad ,$$

$$C = \frac{2\Delta x}{(\Delta x)^2} \quad .$$

Because this formula is not very practical in actual use, (4.28) will be chosen as a guideline for selecting the time step of the method.

E. REFINEMENT OF PARTITION AND EXTRAPOLATION TECHNIQUES

The partition of the domain covered has a great influence on the accuracy of the solution obtained. The choice of grid points is determined by knowledge of the problem and by numerical experimentation. Here, two ways of improving the accuracy of the finite difference method are presented, and one of the two is chosen to be implemented in the numerical methods.

One way to improve the accuracy is the so called prolongation. We first solve the problem using one spacing and then refine the partition and then repeat the computation. If the comparison shows large differences, the process is repeated for smaller and smaller grid sizes until a desired accuracy is achieved. This method may result in a prolonged computational time for the solution.

The second way is called extrapolation. The simple ingenious idea of the technique, which dates back to Richardson in 1910, is the following:

One solves the same type of problem over a prescribed interval, for example [0,1], several times with successively smaller step sizes. Thus, one obtains a sequence of approximations

$$\gamma(1,h_0)\ ,\ \gamma(1,h_1)\ ,\ \ldots$$

for a given sequence of step sizes

$$h_0 > h_1 > \ldots > 0\ .$$

The successive step size h_i is often defined in terms of an input step size h by

$$h_i = \frac{h}{n_i}\ ,\quad i = 0,1,2,\ \ldots\ . \tag{4.31}$$

Thus, any step size sequence $\{h_i\}$ can be characterized by the associated integer sequence $\{n_i\}$. The following are some examples of integer sequences:

$\{1,2,4,8,16,32,\ldots\}$ (Romberg sequence)

$\{1,2,4,6,8,12,\ldots\}$ (Bulirsch sequence)

$\{1,2,3,4,\ldots\}$ (harmonic sequence)

So, the numerical solution at x is computed for a sequence of step size h_i and denoted by $T_{i,0} = \gamma(1,h_i)$. Then, the extrapolation tableau,

$$\begin{array}{l} T_{00} \\ T_{10}, T_{11} \\ T_{20}, T_{21}, T_{22} \end{array}$$

is calculated for x according to two types of commonly used extrapolation schemes

a). Aitken-Neville algorithm

For i = 1,2,... and k = 1,2,3,...i

$$T_{i,k} = \frac{T_{i,k-1} + T_{i,k+1} - T_{i-1,k-1}}{\left(\frac{n_i}{n_{i-k}}\right)^2 - 1} \tag{4.32}$$

b). Rotational extrapolation

For i = 1,2,... and k = 1,2,3,...,i

$$T_{i,-1} = 0 \quad ,$$

$$T_{i,k} = T_{i,k-1} + \frac{T_{i,k-1} - T_{i-1,k-1}}{\left(\frac{n_i}{n_{i-k}}\right)^2\left[1 - \frac{T_{i,k-1} - T_{i-1,k-1}}{T_{i,k-1} - T_{i-1,k-2}}\right] - 1} \quad . \tag{4.33}$$

In this study, extrapolation scheme (4.32) with the Romberg sequence will be implemented when the finite difference method is used to find the numerical solution of the stated problems. It should be noted that if this extrapolation scheme is used the computational time will increase exorbitantly.

V. NUMERICAL RESULTS

A. INTRODUCTION

The problem described in section 1(B) was solved numerically for two special cases, namely, the flat plate ($\alpha_1=1$, $\alpha_2=0$, $\alpha_3=-1$, h=1) and the sphere ($\alpha_1=0$, $\alpha_2=-1$, $\alpha_3=1$, h=1). Since a lot of numerical results of the problem computed by successive approximations method are available in some of the papers[3,4], in this thesis, only the Runge-Kutta method and the finite difference method are employed to the problem for study. Programs are written in Fortran 77 using the Amdahl 5990 model 500 mainframe computer and are set up to allow input for the time step. Thus one can approximate the maximum time step that can be used in a particular numerical method. All calculations are done using double precision arithmetic yielding 12-digit accuracy. Numerical results generated by the methods are compared and discussed.

The Runge-Kutta and the finite difference methods are implemented to solve both special cases. In particular, three different order approximations of the Runge-Kutta method are programmed to solve the integral equations derived by the Laplace transform method. Inefficiency of a high order Runge-Kutta method motivates the use of the finite difference technique. Again, the method is implemented in both cases for various time steps. Some of the numerical results are tabulated and plotted in such a way that a comparison can be made. Notice that the Runge-Kutta method is not applied to the integral equations obtained by the eigenvalue expansion method. The reason is the lag parts of those integral equations diverge when time is zero, and thus, the initial values of the nonlinear ordinary differential equations cannot be computed.

B. RESULTS FOR THE FLAT PLATE AND THE SPHERE

Integral equations (1.39) and (1.43) are solved using the Runge-Kutta method of orders 1, 3, and 5. The first order approximation can be found in [5], whereas the third and the fifth order approximations are described in sections 3(C) and 3(D), respectively. Solutions of the nonlinear ordinary differential equations corresponding to (1.39) and (1.43) are obtained using the fourth-order Runge-Kutta method developed by Zurmuhl [15]. The results show that solutions of a high order approximation fall below those of a lower order approximation (Fig. 5.1, Fig. 5.2, Fig 5.3).

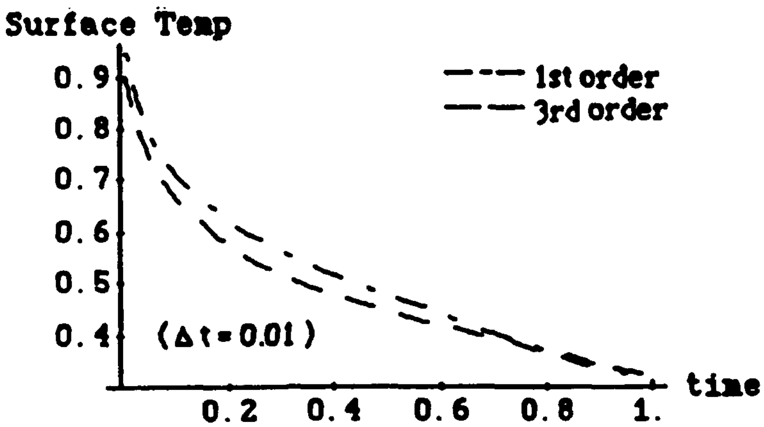

Fig. 5.1 Surface temperature of a Flat Plate cooled by convection and radiation (Runge-Kutta Method).

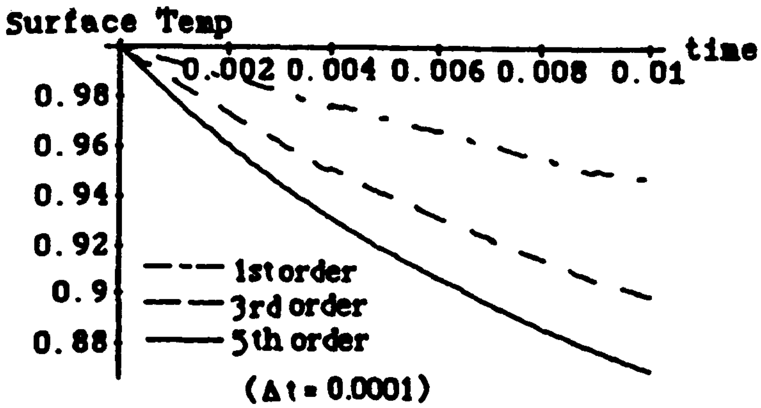

Fig. 5.2 Surface temperature of a Flat Plate cooled by convection and radiation (Runge-Kutta Method).

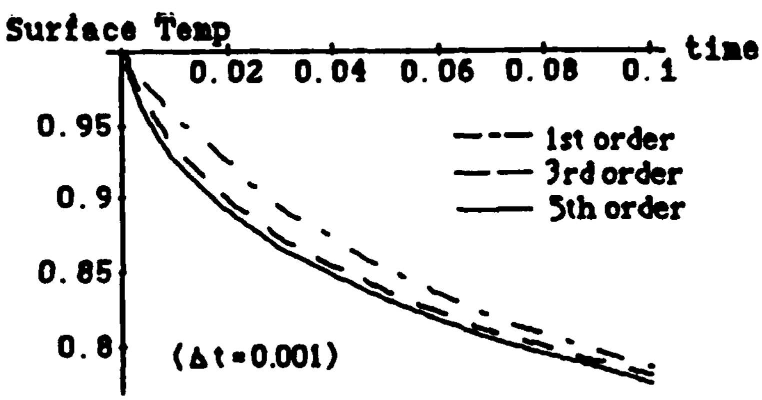

Fig. 5.3 Surface temperature of a sphere cooled by convection and radiation (Runge-Kutta Method).

With respect to time step, we do not have the same phenomenon as in the order of approximation. In a fixed order approximation method, the solution curves for a smaller time step fall below those for a larger time step at small times (approx. less than 0.2) and above at large times (Fig. 5.4).

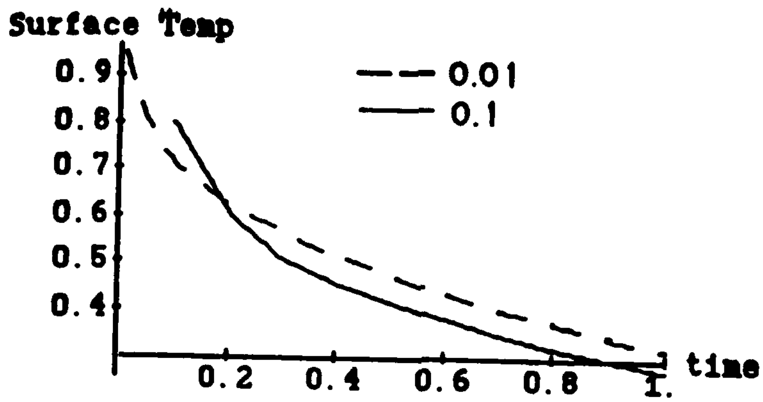

Fig. 5.4 Surface temperature of a Flat Plate cooled by convection and radiation (Runge-Kutta Method of the First Order)

According to numerical experiments, the stability requirements for Runge-Kutta method of orders 1, 3, and 5 are approximately 0.1, 0.01, and 0.001, respectively. As observed earlier, a drawback of the Runge-Kutta method is that it requires the solution of a heuristic nonlinear ordinary differential equation for a high order approximation. This leads to an attempt to use an easier algorithm, and for this reason, the finite difference method was implemented.

Equations (1.1) and (1.2a,b,c) are solved for the flat plate and for the sphere. The extrapolation formula used to improve the accuracy of the solutions is the Aitken-Neville algorithm (4.32). The results for various time steps are presented in tables 1 and 2, and some of these results are plotted in Figures 5.5 and 5.6. As tables 1 and 2 show, the situation where solution curves for a smaller time step fall below those for a larger time step holds in the finite difference method.

Table 1

The Finite Difference Method for Various Time Steps

Time	Δt		
	10^{-2}	10^{-3}	10^{-4}
0.01	0.849395	0.843059	0.842539
0.02	0.797214	0.793829	0.793609
0.03	0.764701	0.762365	0.762256
0.04	0.740419	0.738609	0.738559
0.05	0.720827	0.719340	0.719328
0.06	0.704316	0.703048	0.703063
0.07	0.689998	0.688893	0.688927
0.08	0.677333	0.676351	0.676400
0.09	0.659595	0.665075	0.665137
0.10	0.655626	0.654821	0.654893
0.20	0.583564	0.583127	
0.30	0.535891	0.535564	
0.40	0.496573	0.496292	
0.50	0.461310	0.461056	
0.60	0.428810	0.428575	
0.70	0.398614	0.398394	
0.80	0.370501	0.370296	
0.90	0.344321	0.344130	
1.00	0.319948	0.319770	

The Surface Temperature of a Flat Plate cooled by Convection and Radiation.

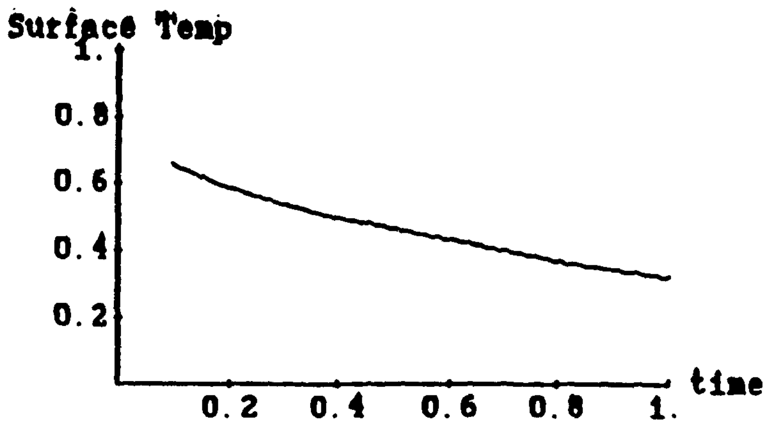

Fig. 5.5 Surface temperature of a flat plate cooled by convection and radiation (Finite Difference Method, Δt=0.01)

Table 2

The Finite Difference Method for Various Time Steps

Time	Δt		
	10^{-2}	10^{-3}	10^{-4}
0.01	0.916249	0.912694	0.913001
0.02	0.882656	0.880526	
0.03	0.860270	0.858708	
0.04	0.842779	0.841514	
0.05	0.828134	0.827055	
0.06	0.815380	0.814431	
0.07	0.803988	0.803134	
0.08	0.793626	0.792846	
0.09	0.784075	0.774505	
0.10	0.775178		
0.20	0.706513		
0.30	0.656561		
0.40	0.616906		
0.50	0.584412		
0.60	0.557215		
0.70	0.534052		
0.80	0.514033		
0.90	0.496514		
1.00	0.481017		

The Surface Temperature of a sphere cooled by Convection and Radiation.

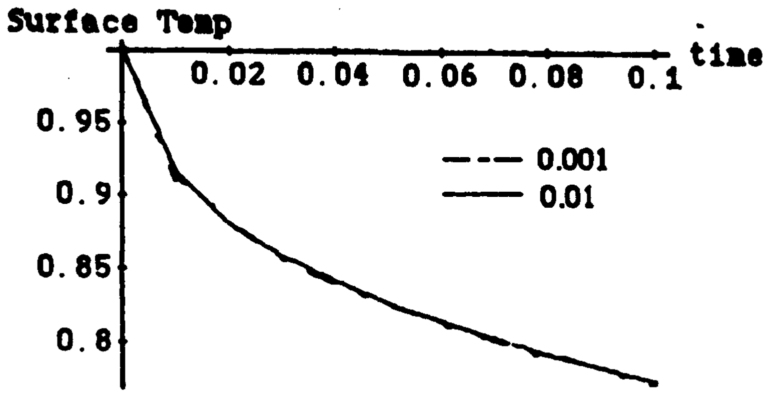

Fig. 5.6 Surface temperature of a sphere cooled by convection and radiation (Finite Difference Method).

Even though the implicit scheme is implemented on the boundary, numerical experiments show that solutions still exhibit oscillation when a large time step was chosen (time step > 0.01). This constraint of time step leads to large computational times for large time solutions.

Figures 5.7 and 5.8 show representative results for the Runge-Kutta and finite difference methods where a flat plate and sphere are cooling. Tables 3 and 4 show that, when Δt = 0.01, the results obtained by using the Runge-Kutta method of orders 2 and 3 compared favourably with those using the finite difference method. The difference of the solutions by using the two methods is less than 3.1% (relative error) in average for each case.

Table 3

Comparison of the Runge-Kutta and the Finite Difference Methods (Δt=0.01)

Time	1st Order	3rd Order	Finite Diff.
0.10	0.712925	0.667017	0.655626
0.20	0.619477	0.567586	0.583564
0.30	0.562454	0.514915	0.535891
0.40	0.516209	0.477480	0.496573
0.50	0.475091	0.445843	0.461310
0.60	0.437583	0.417183	0.428810
0.70	0.403124	0.390595	0.398614
0.80	0.371405	0.365742	0.370501
0.90	0.342189	0.342459	0.344321
1.00	0.315274	0.320636	0.319948

The Surface Temperature of a Flat Plate cooled by Convection and Radiation.

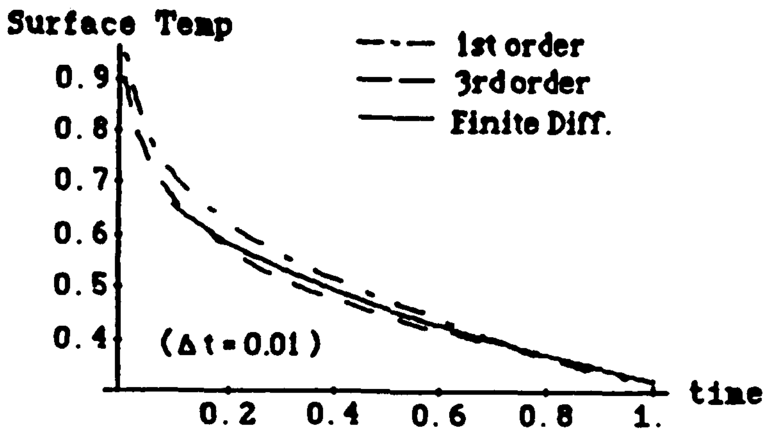

Fig. 5.7 Comparsion of results for cooling a flat plate.

Table 4

Comparison of the Runge-Kutta and the Finite Difference Methods (Δt=0.01)

Time	1st Order	3rd Order	Finite Diff.
0.10	0.785102	0.754306	0.775178
0.20	0.705908	0.665482	0.706513
0.30	0.651471	0.622430	0.656562
0.40	0.609637	0.591087	0.616906
0.50	0.576138	0.565188	0.584412
0.60	0.548557	0.542988	0.557215
0.70	0.525344	0.523640	0.534052
0.80	0.505453	0.506582	0.514033
0.90	0.488156	0.491399	0.496514
1.00	0.472926	0.477772	0.481017

The Surface Temperature of a Sphere cooled by Convection and Radiation.

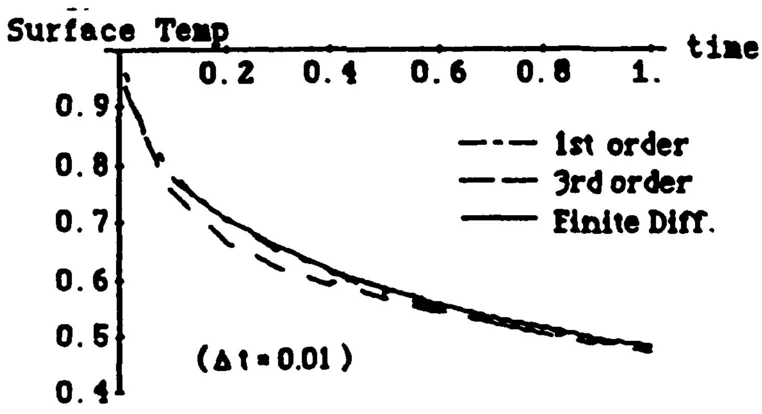

Fig. 5.8 Comparsion of results for cooling of a sphere.

Lastly, in Figures 5.9 and 5.10, the results of two special cases solved by finite difference method are compared. The graph shows that, when $\Delta t = 0.01$, the surface temperature of a flat plate fell much faster than that of a sphere. We believe that it is due to the effect of the boundary condition at $x = 0$ and the difference in the coefficient of the convective term.

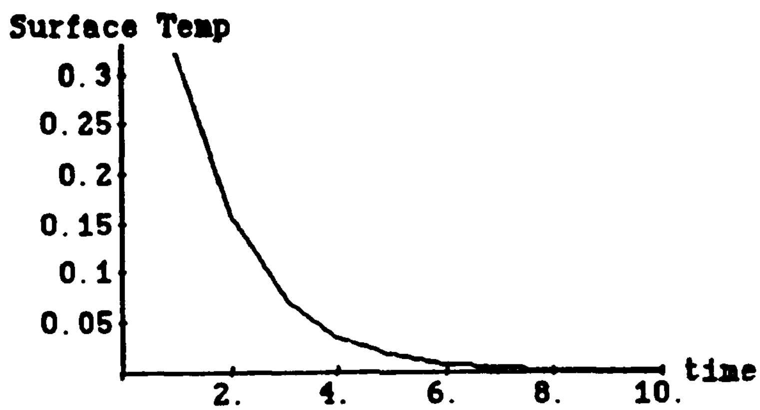

Fig. 5.9 Surface temperature of a flat plate cooled by convection and radiation (Finite Difference Method, $\Delta t=0.01$)

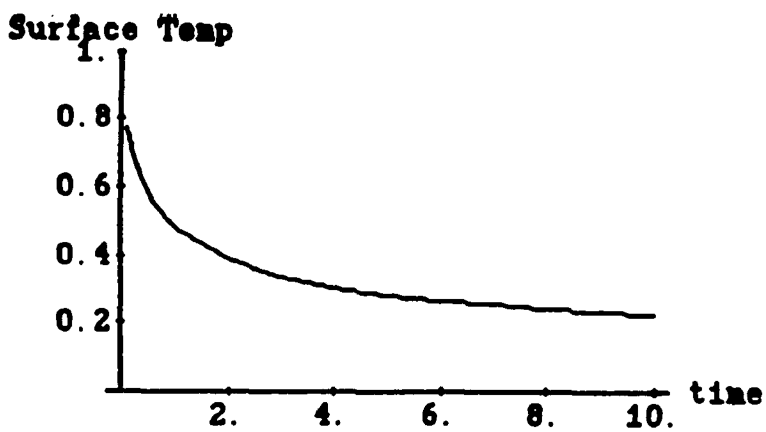

Fig. 5.10 Surface temperature of a sphere cooled by convection and radiation (Finite Difference Method, $\Delta t=0.01$)

VI. CONCLUSIONS

The study of the one dimensional heat equation subject to combined convective and radiative boundary conditions in rectangular coordinates is motivated by the advent of space technology where knowledge of the temperature of bodies in deep space is necessary, for instance, in the design of space shuttles.

The solids are assumed to be homogeneous, isotropic, and opaque to thermal radiation and to have temperature independent physical properties. This assumption leads to a linear heat equation. The difficulty of the problem is determined by the conditions prescribed at the boundaries. According to the laws of physics, the heat flux of the radiative heat transfer is proportional to the fourth power of the temperature which causes nonlinearity at the boundaries.

Problems of this type are first solved by analytic techniques, one of which is the integral transform method. In particular, Laplace transform and eigenvalue expansion are used. The solutions which are explicitly determined at the surface for two special cases, namely, the flat plate and the sphere, are singular nonlinear Volterra integral equations of the second kind. Although they are not practical in determining the temperature at a particular time, these integral equations can help us to deduce some useful information about the behaviour of the surface temperature.

Since the analytic solutions found for the problem are not practical to use, numerical techniques are considered as an alternative. Two numerical schemes that are used to deal with the resulting integral equation are the Runge-Kutta method and the successive approximations method. Both techniques are studied in great detail. Numerical solutions show that the successive approximations method is "exact" in the sense that any desired accuracy may be obtained [3,4]. Additionally, the closer the initial approximation was to the exact solution, the faster the method of successive approximation converged to the exact solution. Conditions for the numerical solution and limitations of these schemes are also discussed.

Another numerical technique which is directly applied to the governing equations is presented as a possible alternative to the numerical methods previously discussed. It is the well known finite difference method in which the Crank- Nicholson scheme, the backward implicit scheme, and the Newton-Raphson method are combined to solve for the surface temperature.

The Runge-Kutta methods of orders 1, 3, and 5 are programmed for (1.39) and (1.43) which are the integral equations corresponding to the flat plate and the sphere, respectively. The numerical results are presented with respect to their orders and to their time steps. The data reveal the following phenomena. First, the solutions of a high order approximation fall below those of a lower order approximation. This phenomena is a result of the higher order approximations closing in on the actual solution. Second, the first phenomenon does not occur in the solutions for various time steps with a fixed approximation order. The main result here is that a smaller step size determines the surface temperature for very small times ($0 \le t \le 0.2$) more accurately and a larger step size determines the surface temperature for larger times ($t \ge 0.2$) more accurately. Third, the agreement between the 1st, 3rd, and 5th order Runge-Kutta approximations is better for the sphere than that for the plate. Physically, this is due to the fact that the boundary surface area to total volume ratio is largest for the sphere and smallest for the plate. The reason for this trend is that the larger the ratio the more uniform will the temperature be throughout the body. The fourth phenomenon is that the accuracy of the approximation increases with time. For large values of time, the rate of change of temperature is reduced, as would be expected from the influence of the fourth power term (U^4). Since the Runge-Kutta method did not offer any efficiency in the area of high order approximations, the finite difference method was considered.

Equations (1.1) and (1.2a,b,c) are solved numerically using the finite difference method for both the flat plate and the sphere. The results for various time steps are presented. The table shows that the second phenomenon found in the Runge-

Kutta method again occurs in the solutions generated by the finite difference method with respect to time step. Again, as in the Runge-Kutta method the smaller step size determines the surface temperature more accurately for small times and the larger step size determines the surface temperature more accurately for large times.

Finally, two comparisons are made of the numerical solutions. The first is of the Runge-Kutta method and the finite difference method. The results show that there is a good agreement between the two methods, and the difference between their solutions are, on the average, less than 3.1% in both cases. The second comparison was made between the solution of a flat plate and that of a sphere. The finite difference method conveys that temperature of a flat plate decays much faster than that of a sphere. This result was expected for the transient heat conduction with linear boundary conditions. This could be due to a larger area on the plate exposed to the uniform boundary layer. Additionally, this result could be caused by the sphere having a larger surface area to volume ratio; thus the sphere would have a more uniform temperature distribution throughout the body resulting in a slower decay of surface temperature.

Comments of a more general nature are included.

1. The convection mode of heat transfer appears to be dominant as the dimensionless temperature approaches uniformity for a plate cooling to a zero environment. This

result is due to the fact that U^4 is approaching zero at much a faster rate then U.

2. Physically, the adiabatic or initial temperature cannot be equal to absolute zero, however, in many situations the temperature ratio of adiabatic surface temperature to initial temperature can be very small.

3. For cooling and heating the solutions are initially inaccurate due to the fact that at t=0 the linearized heat flux is not equal to the actual flux.

4. For a set time step size the number of iterations required to meet a set accuracy is determined by which surface is receiving the highest heat rate.

5. The time required to achieve a particular surface temperature during cooling decreases as the ratio of the environment temperature to initial solid temperature increases.

To conclude this thesis, a numerical scheme is proposed as an alternative to the existing numerical methods. The method of successive approximations is described in Chapter II. One of the major difficulties of that method is choosing the initial approximation for the iteration procedure. As mentioned earlier, the convergence of the algorithm can be accelerated if one could obtain an initial approximation which is close to the exact solution. To determine this value, one could first use the finite difference method (without the extrapolation algorithm) described in Chapter IV to determine the surface temperature. Then, by treating it as an initial approximation, the method of successive approximations is applied to obtain the solution. We believe that the temperature obtained by using the finite difference approximation for the exact solution would be a better solution than the temperature at the previous time level. In addition, this technique would allow larger time steps. However, as far as the author is aware, nothing has been proved for this method, and the analytical and numerical justifications for the algorithm are left open.

APPENDIX-A

To provide a better understanding of the results for $I_1(t)$ and $I_k(t)$ corresponding to (3.29) and (3.30) respectively, consider the following example where N = 3 in (3.17):

$$\begin{bmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \beta_3\alpha_3^2 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & -\beta_3\alpha_3^4 \\ \beta_1\alpha_1^6 & \beta_2\alpha_2^6 & \beta_3\alpha_3^6 \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \\ I_3(t) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{A.1}$$

where

$$A = \begin{bmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \beta_3\alpha_3^2 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & -\beta_3\alpha_3^4 \\ \beta_1\alpha_1^6 & \beta_2\alpha_2^6 & \beta_3\alpha_3^6 \end{bmatrix}. \tag{A.2}$$

Using Cramer's rule

$$I_1(t) = \frac{\begin{vmatrix} B_1 & \beta_2\alpha_2^2 & \beta_3\alpha_3^2 \\ B_2 & -\beta_2\alpha_2^4 & -\beta_3\alpha_3^4 \\ B_3 & \beta_2\alpha_2^6 & \beta_3\alpha_3^6 \end{vmatrix}}{Det(A)} \tag{A.3}$$

where

$$Det(A) = \begin{vmatrix} \beta_1\alpha_1^2 & \beta_2\alpha_2^2 & \beta_3\alpha_3^2 \\ -\beta_1\alpha_1^4 & -\beta_2\alpha_2^4 & -\beta_3\alpha_3^4 \\ \beta_1\alpha_1^6 & \beta_2\alpha_2^6 & \beta_3\alpha_3^6 \end{vmatrix} . \tag{A.4}$$

Using a fundamental property of determinants (A.4) can be written as

$$Det(A) = -\beta_1\beta_2\beta_3(\alpha_1^2\alpha_2^2\alpha_2^3)\begin{vmatrix} 1 & 1 & 1 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 \end{vmatrix} . \tag{A.5}$$

Column reduction gives

$$Det(A) = -\beta_1\beta_2\beta_3(\alpha_1^2\alpha_2^2\alpha_3^2)\begin{vmatrix} 1 & 0 & 0 \\ \alpha_2^2 & -(\alpha_2^2-\alpha_1^2) & -(\alpha_3^2-\alpha_1^2) \\ \alpha_1^4 & (\alpha_2^4-\alpha_1^4) & (\alpha_3^4-\alpha_1^4) \end{vmatrix} . \tag{A.6}$$

Define

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ \alpha_2^2 & -(\alpha_2^2-\alpha_1^2) & -(\alpha_3^2-\alpha_1^2) \\ \alpha_1^4 & (\alpha_2^4-\alpha_1^4) & (\alpha_3^4-\alpha_1^4) \end{bmatrix} . \tag{A.7}$$

then

$$Det(A') = \begin{vmatrix} -(\alpha_2^2-\alpha_1^2) & -(\alpha_3^2-\alpha_1^2) \\ (\alpha_2^4-\alpha_1^4) & (\alpha_3^4-\alpha_1^4) \end{vmatrix} . \tag{A.8}$$

By a similar manipulation (A.3) becomes

$$I_1(t) = \frac{\begin{vmatrix} (B_2+\alpha_2^2 B_1) & -(\alpha_3^2-\alpha_2^2) \\ (B_3-\alpha_2^4 B_1) & (\alpha_3^4-\alpha_2^4) \end{vmatrix}}{\beta_1\alpha_1^2 \, Det(A')} . \tag{A.9}$$

After working through a considerable amount of algebra both (A.3) and (A.9) give

$$I_1(t) = \frac{B_1\alpha_2^2\alpha_3^2 + B_2(\alpha_3^2 + \alpha_2^2) + B_3}{\beta_1\alpha_1^2[\alpha_2^2\alpha_3^2 - \alpha_1^2(\alpha_3^2 + \alpha_2^2) + \alpha_1^4]} .$$

In a similar manner $I_2(t)$ and $I_3(t)$ can be determined. Without loss of generality $I_k(t)$ can also be found.

APPENDIX-B

The Thomas Algorithm

The equations are:

$$a_i U_{i-1} + b_i U_i + c_i U_{i+1} = d_i \quad ,$$

where $1 \le i \le N$ with $a_1 = c_1 = 0$, and N is the number of nodes in the domain.

The algorithm is as follows:

```
      DO 10 i = 2,N
        ratio = a_i/b_i
           b_i = b_i - ratio * c_(i-1)
           d_i = d_i - ratio * d_(i-1)
   10 CONTINUE
           U_N = d_N/b_N
      DO 20 i = N-1,1
           U_i = (d_i - c_i * d_(i+1))/b_i
   20 CONTINUE
```

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Your participation in this research and response by 12 March 1993 is greatly appreciated. If you need any additional details, contact LCDR Richard Mendez (408) 759-9783/LT Gerald Rivas (408) 655-1625, or by writing to:

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Monterey, CA 93943-5000

Name: ____

Address: ____

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ALLOTMENT

An authorization by the head (or other authorized employee) of an operating agency which assigns a specified amount of money to subordinate units. The amount allotted by the agency cannot exceed the amount apportioned by the Office of Management and Budget (OMB).

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1	--------2	----------3	----------4	----------5	---------6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

____

____

____

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Synonyms: ____

Antonyms: ____

AWARD

(1) The formal acceptance of an offeror's bid or proposal.
(2) Notification of intent to give a contract.
(3) Transmittal of advance authorization to proceed (e.g. letter contract).

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1--------	-2----------	-3----------	-4----------	-5----------	-6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

CERTIFICATION

The formal act of acknowledging in writing and affirming by signature that:

- some act has or has not been performed;
- some event has or has not occurred;
- some legal formality has or has not been complied with; or
- some condition exists or does not exist.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1--------2----------3----------4----------5---------6----

STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

CONSULTANT

A person having specialized education and/or broad experience which uniquely qualifies them to be called upon to furnish expert advice on highly specialized matters and recommend solutions to particular problems.

Synonyms: Advisor, Expert, Subject Matter Expert (SME)
Antonyms: None

Do you agree with this definition?

-----1--------2----------3----------4----------5---------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

COST OBJECTIVE

(1) A function, organizational subdivision, contract, or other work unit for which cost data is desired and for which provision is made to accumulate and measure the cost of processes, products, capitalized projects, and so forth.
(2) Cost goal established for the completion of an element of work.
(3) Goal established for contract cost to be achieved during contract negotiations.

Synonyms: Cost Center, Cost Goal, Target Cost
Antonyms: None

Do you agree with this definition?

-----1--------	2---------	3---------	4---------	5---------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ____

Antonyms: ____

ESCALATION

(1) A term traditionally used to indicate an upward or (more rarely) a downward movement of price. "Economic Price Adjustment" is the contemporary term used to express the application of escalation by specified procedures.

(2) In Government contracting refers to an amount or percent by which a contract price may be adjusted if predefined contingencies occur, such as changes in the vendor's raw material costs or labor costs. The amount of the "escalation" is usually tied to some predetermined price index.

Synonyms: Economic Price Adjustment
Antonyms: None

Do you agree with this definition?

-----1--------2----------3----------4----------5---------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

EXPENSE

Costs of operation and maintenance of activities on the accrual basis for a fiscal period, as distinguished from capital costs that will be depreciated over their approximate service life.

Synonyms: Costs
Antonyms: Revenue, Income

Do you agree with this definition?

-----1	--------2	----------3	----------4	----------5	---------6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

INDUSTRIAL PLANT EQUIPMENT (IPE)

Plant equipment acquired by the Government, exceeding an established acquisition cost threshold, used for the purpose of cutting, abrading, grinding, shaping, forming, joining, testing, measuring, heating, treating or otherwise altering the physical, electrical or chemical properties of materials, components or other end items entailed in manufacturing, maintenance, supply, processing, assembly or research and development operations.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1--------2----------3----------4----------5---------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ____

Antonyms: ____

OFFSETS

A cost balancing action whereby a claim may be canceled or lessened by a counterclaim.

Defective pricing: Allowable understatements (e.g., counterclaims or cost proposal errors that are favorable to the contractor) which are reduced by overstatements of cost that arise under a defective pricing case. In order to eliminate an increase in the contract price the offset cannot exceed the extent of the overstatement.

Administrative Offset: A procedure to collect a debt owed to the Government by withholding money payable to contractor under a contract, in order to satisfy the contractor's debt which arose independently of that contract and which are in compliance with the Federal Claims Collection Act of 1966.

Synonyms: Counterclaim, Setoff
Antonyms: None

Do you agree with this definition?

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

WEIGHTED AVERAGE COST METHOD

A method of determining the average unit cost of inventory and by implication an aid in determining the cost of goods made, sold, or held for future sale or incorporation into higher level end items. Under this technique, costs are periodically computed by adding the sum of the costs of beginning inventory with the sum of the costs of subsequent purchases and dividing by the total number of units.

Synonyms: None
Antonyms: None

Do you agree with this definition?

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

Allocation

Funding: An amount of money transferred from one agency, bureau or account that is set aside in an appropriation of the various committees having spending responsibilities to carry out the purposes of the parent appropriation or fund.

Within DOD, the money is being transferred from the services to the appropriate MAJCOMS.

Financial: A cost accounting procedure which results in a reasonable distribution of costs among one or more cost objectives (e.g., products, programs, contracts, and activities). This includes both direct assignment of costs and the reassignment of a share from an indirect pool.

Synonyms: None
Antonyms: None

Do you agree with this definition?

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ____

Antonyms: ____

Amortization

The systematic reduction of an indebtedness or recorded asset value over a specific period of time by periodic payments to a creditor or charges to an expense, in accordance with generally accepted accounting procedures or principles.

Synonyms: Liquidation, Allocation, Writeoff
Antonyms: Direct charge

Do you agree with this definition?

-----1--------2----------3----------4----------5---------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

Bid

An offer to perform a contract by providing labor and or material for a specific price. In federal government contracting, this offer is provided in response to an invitation for bid.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1--------2----------3----------4----------5---------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

Commitment

The act by an authorized individual affirming the intent of an agency or company to take or accept a defined action not yet formalized by execution of a contract.

Funding: A firm administrative reservation of funds based upon firm procurement directions, orders, requisitions, certified purchase requests, and budgetary authorizations which set aside certain funds for a particular contract without further recourse to the official responsible for certifying the availability of funds.

Within DOD, reservation of funds are set aside by the appropriate operating division (wing or base) for use on a particular item.

Accounting: The method of accounting for the available balance of an appropriation, fund, or contract authorization whereby commitments are recorded in the accounts as reductions of the available balance.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1	--------2	---------3	---------4	---------5	---------6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

Cost

(1) For the Seller: The amount of money or equivalent incurred for supplies or services exclusive of profit or fee.

(2) For the Buyer: The amount of money or equivalent paid for supplies or services including profit or fee.

Synonyms: Expense, Consideration, Charge, Total Cost
Antonyms: None

Do you agree with this definition?

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

Synonyms:

Antonyms:

Delinquency

(1) Failure, omission, or violation of contractual obligation or duty.
(2) The actual failure by the contractor to meet the contract delivery or performance schedule, or the potential failure to do so by failing to maintain required progress in contract performance as required by the contract delivery or performance schedule

Synonyms: Overdue, Tardy, Late
Antonyms: Early, Accelerated, Timely

Do you agree with this definition?

-----1	--------2	---------3	---------4	---------5	---------6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

Executed Contract

A written document which has been signed by both parties and mailed or otherwise furnished to each party, which expresses the requirements, terms, and conditions to be met by each party.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1--------	2---------	3---------	4---------	5---------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ____

Antonyms: ____

Independent Cost Estimate

A cost estimate developed outside the normal advocacy channels, independent of any cost information provided by the offeror, used for the purpose of comparing with bids or proposals. Preparation of independent costs estimates generally include representations from the areas of cost analysis, procurement, production management, engineering, and program management.

Synonyms: Independent Government Cost Estimate (IGCE)
Antonyms: None

Do you agree with this definition?

-----1--------2---------3---------4---------5--------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ________________________________

Antonyms: ________________________________

Novation Agreement

A legal instrument, executed by the parties to a contract and a successor in interest, which transfers all obligations and rights under the contract to the successor.

The government may recognize a third party as a successor of a government contract when the third party's interests arises out of the transfer of 1) all the contractor's assets, or 2) the entire portion of the assets involved in the performing a contract.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1--------2---------3---------4---------5--------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ______________________________________

Antonyms: ______________________________________

Royalty

Compensation paid to the owner, vendor or lessor of personal, real, tangible or intangible property for the use of that property. Usually a percentage of the selling price of goods and services, production of which employs the property.

Synonyms: Commission Payment, Use Fee
Antonyms: Royalty Free Use

Do you agree with this definition?

-----1--------2---------3---------4---------5---------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT:

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Synonyms: ____

Antonyms: ____

APPENDIX C: FOLLOW-UP QUESTIONNAIRE

This appendix represents the follow-on questionnaire which was mailed to the respondents from the initial questionnaire who provided names and addresses.

A. FOLLOW-ON QUESTIONNAIRE

Thank you for completing the initial questionnaire and participating in this follow-on research to arrive at consensus definitions of contracting terminology. Your efforts have provided an excellent base for the establishment of a consensus. This questionnaire will only be sent to those who responded to the initial questionnaire, so your continued participation is very important.

As a reminder: Graduate students at the Naval Postgraduate School, Monterey, California, and the Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, are conducting research to derive baseline definitions for commonly used acquisition words or phrases. When the project is complete, the definitions will be included as part of a professional dictionary of contracting terminology that will be published by the NCMA. The purpose of the dictionary is two fold. First, to provide an educational tool to those unfamiliar with the acquisition process. Second to provide a reference document for those working in the field. This research is an ongoing effort in obtaining feedback from contracting professionals regarding proposed definitions of contracting terms. It differs from the previous research in that it is taking terms from previous efforts which generated significant diversity, and is refining them using the Delphi Technique. All terms were synthesized from collected definitions, Government regulations and contracting literature and were reviewed once by NCMA Fellows and Certified professionals prior to your input on the initial questionnaire.

Attached for your review are the revised definitions and selected comments from the initial questionaires. The definitions were revised by the researchers and reviewed by a committee of contracting professionals for compliance with the consensus. Please review the revised definitions and indicate your agreement level on the scale provided from 1 to 6. If you have any disagreements or comments, please either annotate them where applicable, or write them on the space provided.

Your continued participation in this research and response by 26 April 1993 is greatly appreciated. If you need any additional details, contact LCDR Richard Mendez or LT Gerald Rivas by telephone at (408) 656-2536 (Administrative Science Curriculum Office), or by writing to:

LT Gerald A. Rivas
SMC #2715
Naval Postgraduate School
Monterey, California 93943-5000

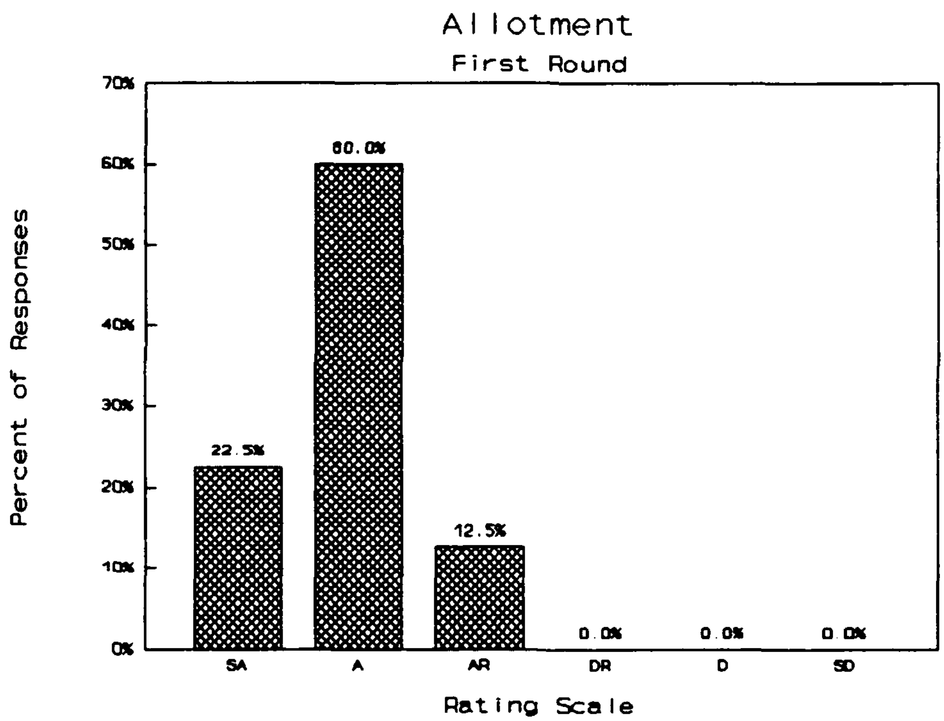
Original Definition:

ALLOTMENT

An authorization by the head (or other authorized employee) of an operating agency which assigns a specified amount of money to subordinate units. The amount allotted by the agency cannot exceed the amount apportioned by the Office of Management and Budget (OMB).

Synonyms: None
Antonyms: None

Survey Results



Comments:

Allotments can be made by other than "heads" to "subordinate units".

Allotments go farther than to subordinate agency units. They end up being made to programs/projects and individual contracts.

Periodicity of allotments, i.e. quarterly/annually.

Add to end of first sentence ", projects or activities."

Can negotiation go on between subordinate units.

Government Contracts Reference Book definition: "In DOD, the process by which commanders, Major Commanders, or Special Operating Agencies distribute their allocated funds to themselves, to installation commanders or to other subordinate organizations. This process may continue into as many sub allotments as necessary."

Synonyms: Funding, Budgeted Amount, Obligation, Appropriation, Public Troth.
Antonyms:

Revised Definition:

ALLOTMENT

An authorization by the head (or other authorized employee) of an operating agency which assigns a specified amount of money to subordinate units, **projects or activities**. The amount allotted by the agency cannot exceed the amount apportioned by the Office of Management and Budget (OMB).

Synonyms: **Funding.**
Antonyms: None

Do you agree with this definition?

-----1------2----------3-----------4----------5----------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

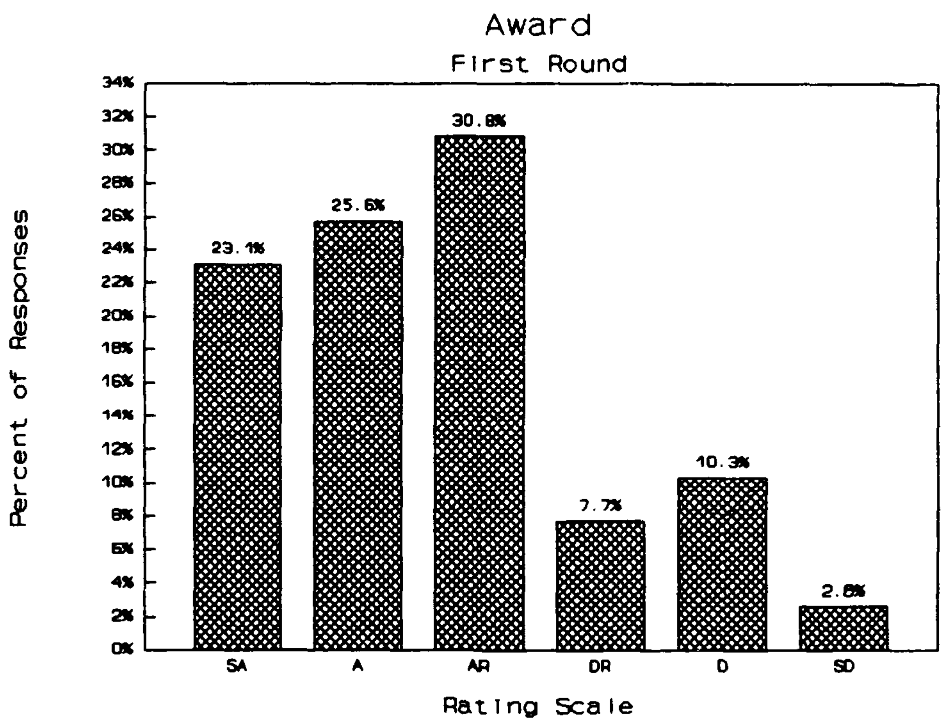
Original Definition:

AWARD

(1) The formal acceptance of an offeror's bid or proposal.
(2) Notification of intent to give a contract.
(3) Transmittal of advance authorization to proceed (e.g. letter contract).

Synonyms: None
Antonyms: None

Survey Results



Comments:

In (2) change "give" to "let".

Very essential to establish clear and concise communication between contractor and Government contracting personnel.

Delete (2). Notification of intent to award does not constitute award. Enhance definition by adding "Notice of Award, Notice to Proceed" to (3).

Award is the process through which a buyer and seller come to agreement over the terms of a contract, "award" is always rendered by the buyer.

Item (2) is too broad - needs more specifics - leaves out FAR/DFAR.

Item (3) is a conditional award.

Item (2) may or may not result in a contract depending on state of negotiation.

"Notification of intent" and "advance authorization" are not considered "award".

Item (1) needs to mention "formal acceptance by an authorized official of the Government".

Item (2), intent is not an award.

Items (2) and (3) are a result of Item (1) and they infer acceptance.

Add to end of (1) "as offered."

Item (2) requires return notice of acceptance by offeror.

Reference Book Definition: "The notification by the Government that it will contract with a private party. The award of a contract is usually made by Acceptance of an Offer that has been made by an offeror. In procurements by sealed bidding, the contracting officer makes a contract award by written notice, within the time for acceptance specified in the bid or extension, to the responsible bidder whose bid, conforming to solicitation, is the most advantageous to the Government, considering only price and price-related factors included in the solicitation. In procurements by negotiation, the contracting officer awards a contract with reasonable promptness to the successful offeror (the source whose best and final offer (BAFO) is most advantageous to the Government considering price and other factors included in the solicitation) by transmitting a written notice of award to that offeror.

Synonyms: Contract, Win, Definitization of Contract.
Antonyms: Loss

Revised Definition:

AWARD

(1) The formal acceptance of an offeror's bid or proposal.
(2) Transmittal of advance authorization to proceed (e.g. letter contract).

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1------2----------3----------4----------5----------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

C O M M E N T :

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SYNONYMS: ______________________________

ANTONYMS: ______________________________

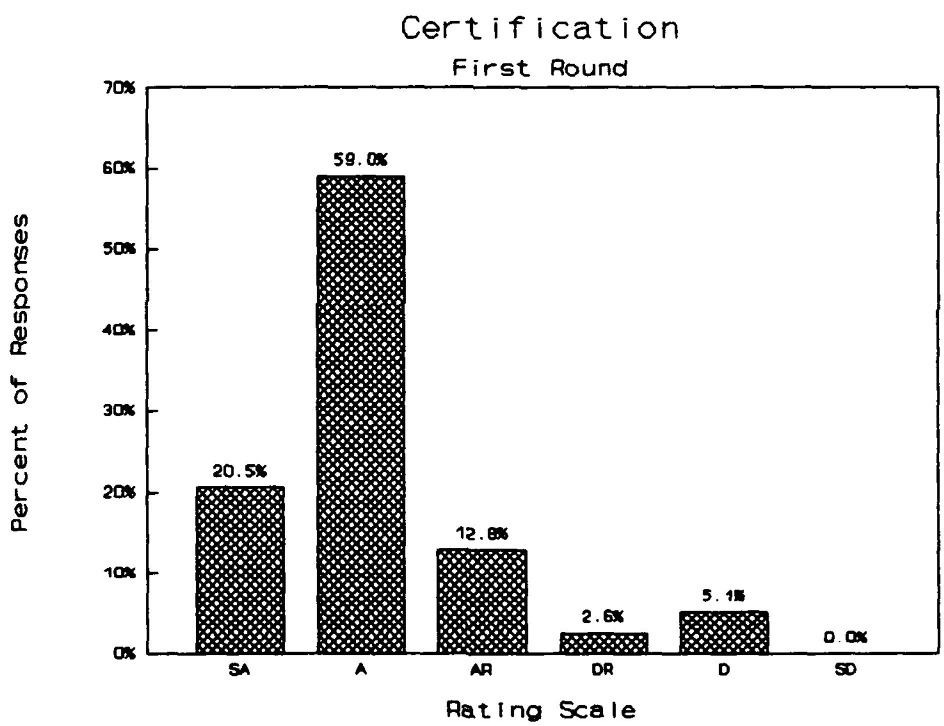
Original Definition:

CERTIFICATION

The formal act of acknowledging in writing and affirming by signature that:
- some act has or has not been performed;
- some event has or has not occurred;
- some legal formality has or has not been complied with; or
- some condition exists or does not exist.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Needs to follow FAR/DFARS/CFR more closely.

Need to mention legal accountability of certifier.

Reword definition as a positive statement. Strike "or has not" and "or does not" from sentences.

Requirement should be mentioned in definition.

Delete item 3.

Synonyms:
Antonyms:

Revised Definition:

CERTIFICATION

The formal act of acknowledging in writing and affirming by signature that:
- some act has or has not been performed;
- some event has or has not occurred;
- some legal formality has or has not been complied with; or
- some condition exists or does not exist.

Synonyms: None
Antonyms: None

Do you agree with this definition?

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

C O M M E N T :

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SYNONYMS: _______________________________________

ANTONYMS: _______________________________________

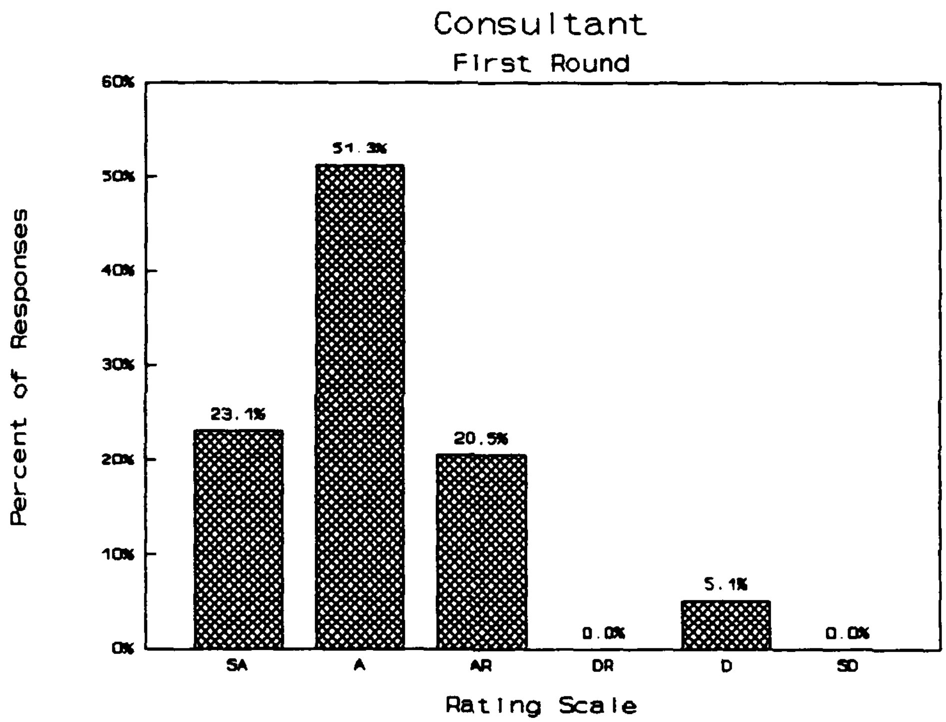
Original Definition:

CONSULTANT

A person having specialized education and/or broad experience which uniquely qualifies them to be called upon to furnish expert advice on highly specialized matters and recommend solutions to particular problems.

Synonyms: Advisor, Expert, Subject Matter Expert (SME)
Antonyms: None

Survey Results



Comments:

Change "them" to "him/her".

Change "broad" to "extensive".

Consultants may be called upon to furnish guidance or to advocate, for the benefit of an individual or entity.

Add after "expert advice" - "or opinions".

Add after "called upon" - "by the Federal Government".

Add to end of definition "of a non-inherently governmental nature."

Definition too narrow. Is deliverable required?

Consultant connotes a business relationship unlike synonyms.

Change "highly specialized" to "various or relevant".

Add to end of definition "and is so called upon for that specific purpose." this will exclude persons already obligated by govt contract.

Synonyms: Specialist, Facilitator, Authority.
Antonyms: Employee.

Revised Definition:

<u>CONSULTANT</u>

A person having specialized education and/or broad experience which uniquely qualifies **him/her** to be called upon to furnish expert advice **or opinions** on highly specialized matters and recommend solutions to particular problems.

Synonyms: Advisor, Expert, Subject Matter Expert (SME), **Specialist, Authority.**
Antonyms: None

Do you agree with this definition?

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

C O M M E N T :

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SYNONYMS: ______________________________

ANTONYMS: ______________________________

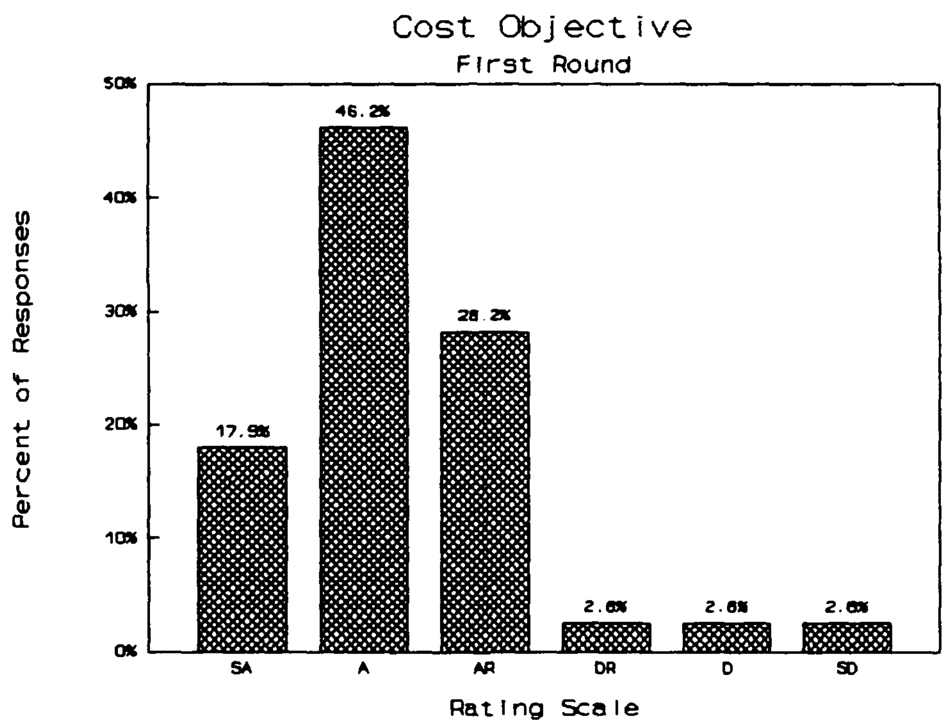
Original Definition:

COST OBJECTIVE

(1) A function, organizational subdivision, contract, or other work unit for which cost data is desired and for which provision is made to accumulate and measure the cost of processes, products, capitalized projects, and so forth.
(2) Cost goal established for the completion of an element of work.
(3) Goal established for contract cost to be achieved during contract negotiations.

Synonyms: Cost Center, Cost Goal, Target Cost
Antonyms: None

Survey Results



Comments:

Delete (1) and "Cost Center" synonym.

Add to (1) after "data is desired" - "and/or required".

Change definition to "Cost objective is a measure of applicable dollars to a defined task/work effort. Can apply to a contract, organization or other work unit."

Synonyms: Cost Segment.
Antonyms: None

Revised Definition:

COST OBJECTIVE

Accounting: A function, contract, or other work unit for which cost data is desired and for which provision is made to accumulate and measure the cost of processes, products, capitalized projects, and so forth.

Program Management: Cost goal established for the completion of an element of work.

Negotiations: Goal established for contract cost to be achieved during contract negotiations.

Synonyms: Cost Goal, Target Cost
Antonyms: None

Do you agree with this definition?

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

C O M M E N T :

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SYNONYMS: ____

ANTONYMS: ____

Original Definition:

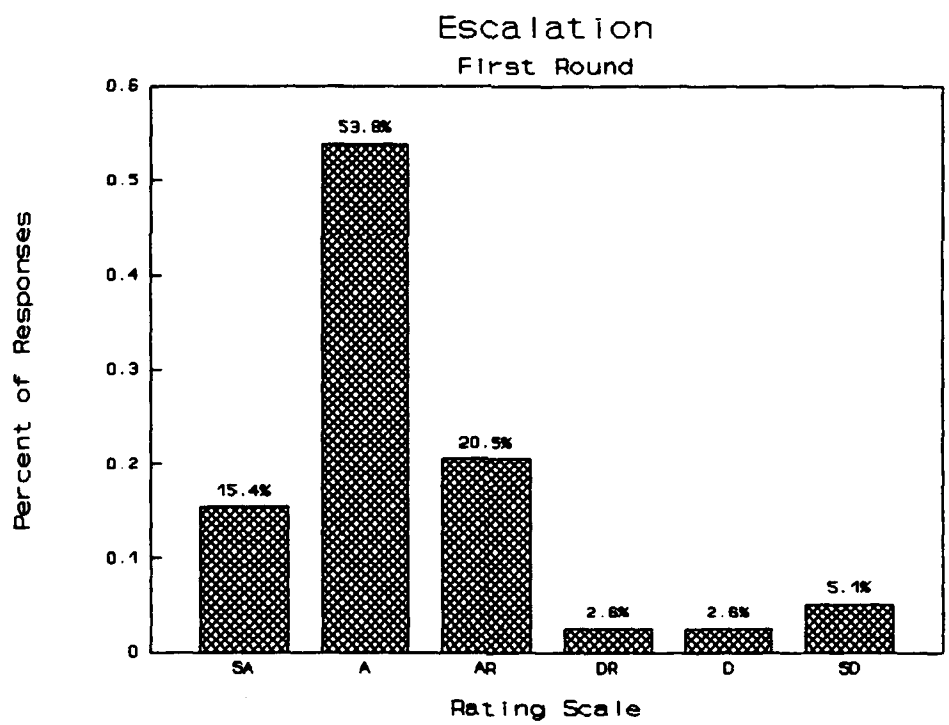
ESCALATION

(1) A term traditionally used to indicate an upward or (more rarely) a downward movement of price. "Economic Price Adjustment" is the contemporary term used to express the application of escalation by specified procedures.

(2) In Government contracting refers to an amount or percent by which a contract price may be adjusted if predefined contingencies occur, such as changes in the vendor's raw material costs or labor costs. The amount of the "escalation" is usually tied to some predetermined price index.

Synonyms: Economic Price Adjustment
Antonyms: None

Survey Results



Comments:

Delete definition, add "A term traditionally used to indicate the periodic price adjustment of a contract. It is frequently computed by a mathematical formula, specified in the contract or BOA, utilizing well known national indices. It is not uncommon for the contractor to be limited to recovery of only a portion of the total fluctuation defined by the formula as part of the risk sharing arrangement of the contract.

Add to (1) "A pricing term".

Change (2) to "In Government contracting refers to an amount, rate or percent by which a contract price may be adjusted if predefined contingencies occur, such as significant changes beyond its control in the vendor's raw material costs or labor costs. The amount of the "escalation" must be tied to some predetermined price index."

Add to end of first sentence in (1) "/cost."

Escalation would be only an upward movement while EPA could go both ways.

Eliminate "more rarely" in (1).

Add to (2) after "such as changes" - "upward or downward".

EPA is not synonym of Escalation. EPA is application of escalation.

Should contract type be included in definition, i.e. CP or FFP?

Change (1) "application of escalation by specified procedures" to "application of previously agreed price adjustment(s) after contract award.

Change (2) "escalation" to "adjustment".

Change end of (2) "predetermined price index" to "predetermined public or Government price index."

Change first sentence in (1) to "A price increase or revision upward due to external influences such as inflation or market adjustments."

Synonyms: Cost Growth.
Antonyms: Deescalation.

Revised Definition:

ESCALATION

A **pricing** term traditionally used to indicate an upward movement of price/**cost due to inflation or market adjustment**.

Synonyms: **None**
Antonyms: **Deescalation**

Do you agree with this definition?

-----1------	2----------	3----	------4----------	5----------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

C O M M E N T :

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SYNONYMS: ________________________________

ANTONYMS: ________________________________

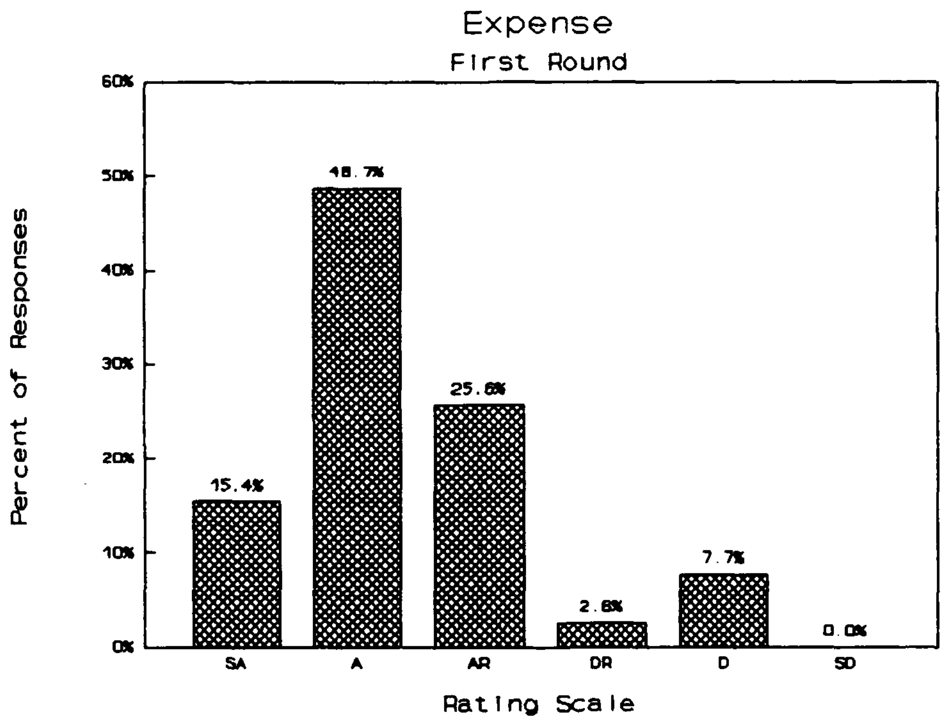
Original Definition:

EXPENSE

Costs of operation and maintenance of activities on the accrual basis for a fiscal period, as distinguished from capital costs that will be depreciated over their approximate service life.

Synonyms: Costs
Antonyms: Revenue, Income

Survey Results



Comments:

Change definition to "A cost incurred in performance of a business operation or contract to be accounted for on an accrual basis for a fiscal year or tied to a specific contract. As distinguished from capital ...."

Delete "Revenue, Income" from antonyms

Change "will be depreciated" to "depreciate".

Could break down definition into cash expenses and non-cash expenses.

Change "on the accrual basis for a fiscal period, as ...." to "for a fiscal period."

Change definition to "The collection of costs related to a particular defined set of activities, over a set period of time."

Change "Costs of operation" to "Reasonable costs, direct and indirect, of operation".

Why only "accrual basis"? Is it true for "cost basis"?

Synonyms: Costs, Burdens, Indirect Costs, Outgo, Overhead Item, Consumption, Spending.
Antonyms: Fee.

Revised Definition:

EXPENSE

Costs of operation and maintenance of activities on the accrual basis for a fiscal period, as distinguished from capital costs that **depreciate** over their approximate service life.

Synonyms: Costs
Antonyms: Revenue, Income

Do you agree with this definition?

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STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

C O M M E N T :
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SYNONYMS: ________________________________

ANTONYMS: ________________________________

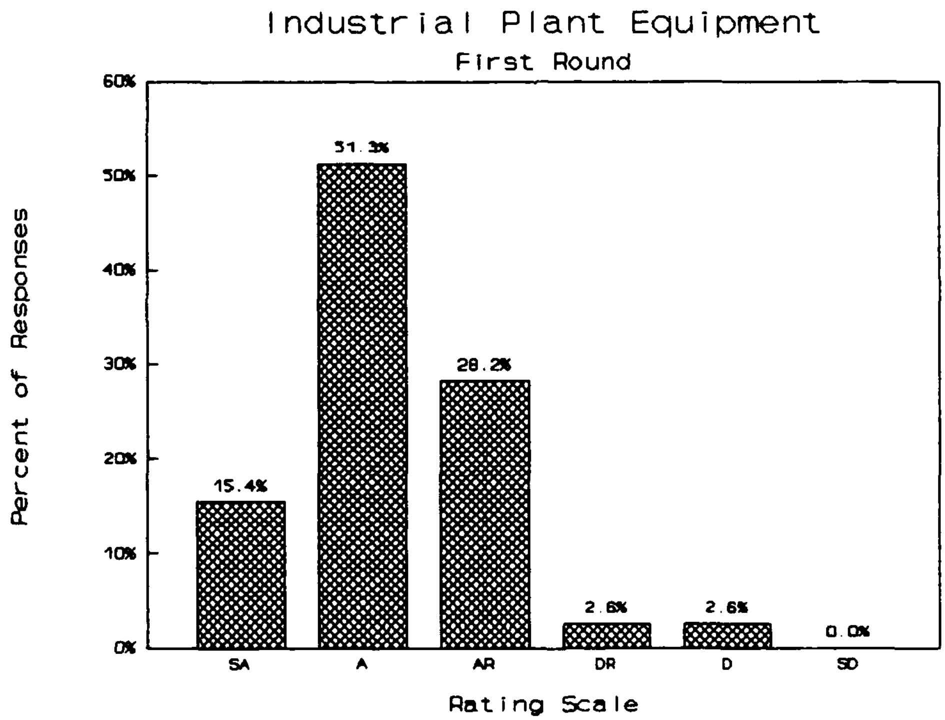
Original Definition:

INDUSTRIAL PLANT EQUIPMENT (IPE)

Plant equipment acquired by the Government, exceeding an established acquisition cost threshold, used for the purpose of cutting, abrading, grinding, shaping, forming, joining, testing, measuring, heating, treating or otherwise altering the physical, electrical or chemical properties of materials, components or other end items entailed in manufacturing, maintenance, supply, processing, assembly or research and development operations.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Add to end of definition "anticipated to have value and use after the contract is completed."

Delete "exceeding an established acquisition cost threshold". IPE is IPE if it is within or in excess of a cost threshold.

DFARS 245.301.

Add to definition "This equipment is Government Furnished Equipment (GFE) for the contractors' use in furtherance of the Government contract."

Delete listing type of equipment - too limiting.

IPE is not necessarily acquired by the Government in all cases. The contractor must sometimes invest in IPE.

Synonyms: None
Antonyms: None

Revised Definition:

INDUSTRIAL PLANT EQUIPMENT (IPE)

Plant equipment acquired by **either** Government **or industry**, exceeding an established acquisition cost threshold, used for the purpose of altering the physical, electrical or chemical properties of materials, components or other end items entailed in manufacturing, maintenance, supply, processing, assembly or research and development operations.

Synonyms: None
Antonyms: None

Do you agree with this definition?

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STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

C O M M E N T :

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SYNONYMS: ______________________________

ANTONYMS: ______________________________

Original Definition:

OFFSETS

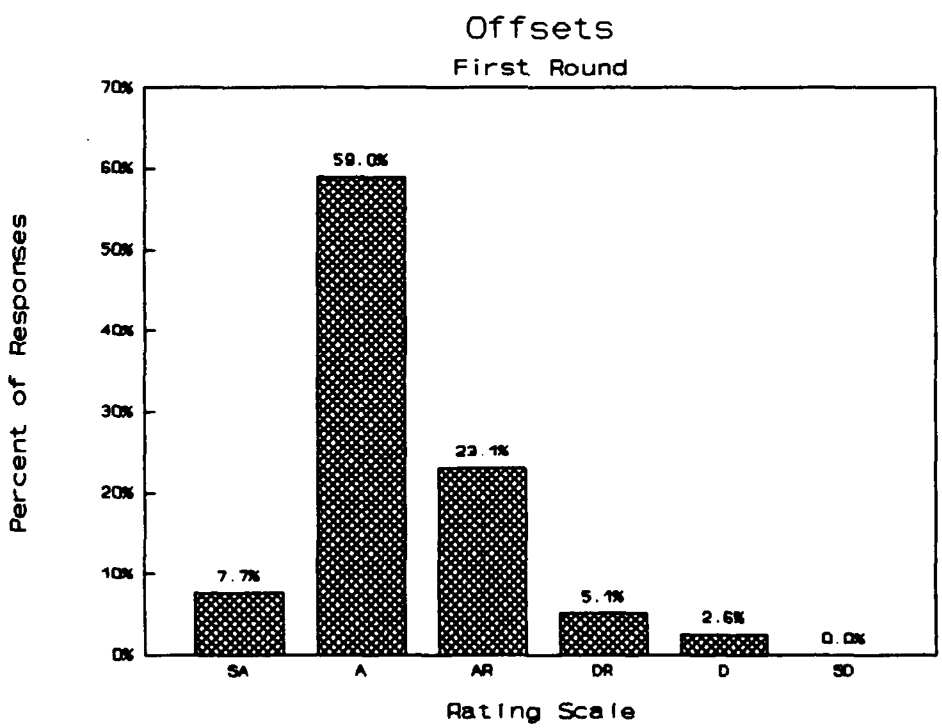
A cost balancing action whereby a claim may be canceled or lessened by a counterclaim.

Defective pricing: Allowable understatements (e.g., counterclaims or cost proposal errors that are favorable to the contractor) which are reduced by overstatements of cost that arise under a defective pricing case. In order to eliminate an increase in the contract price the offset cannot exceed the extent of the overstatement.

Administrative Offset: A procedure to collect a debt owed to the Government by withholding money payable to contractor under a contract, in order to satisfy the contractor's debt which arose independently of that contract and which are in compliance with the Federal Claims Collection Act of 1966.

Synonyms: Counterclaim, Setoff
Antonyms: None

Survey Results



Comments:

International offsets left out.

An offset may be a deduction or credit, as well.

Delete "In order to eliminate an increase in the contract price the offset cannot exceed the extent of the overstatement."

Add to paragraph 3 "payable to the contractor ...".

Add to paragraph 3 "A unilateral procedure ...".

Add to end of paragraph 1 "A tradeoff wherein a cost is allowed for a particular segment of the work but a corresponding reduction in cost is agreed upon for another segment."

Synonyms:
Antonyms:

Revised Definition:

OFFSETS

A cost balancing action whereby a claim may be canceled or lessened by a counterclaim.

Defective pricing: Allowable understatements (e.g., counterclaims or cost proposal errors that are favorable to the contractor) which are reduced by overstatements of cost that arise under a defective pricing case. In order to eliminate an increase in the contract price the offset cannot exceed the extent of the overstatement.

Administrative Offset: A procedure to collect a debt owed to the Government by withholding money payable to the contractor under a contract, in order to satisfy the contractor's debt which arose independently of that contract and which are in compliance with the Federal Claims Collection Act of 1966.

Synonyms: Counterclaim, Setoff
Antonyms: None

Do you agree with this definition?

-----1------2----------3-----------4----------5----------6----

STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

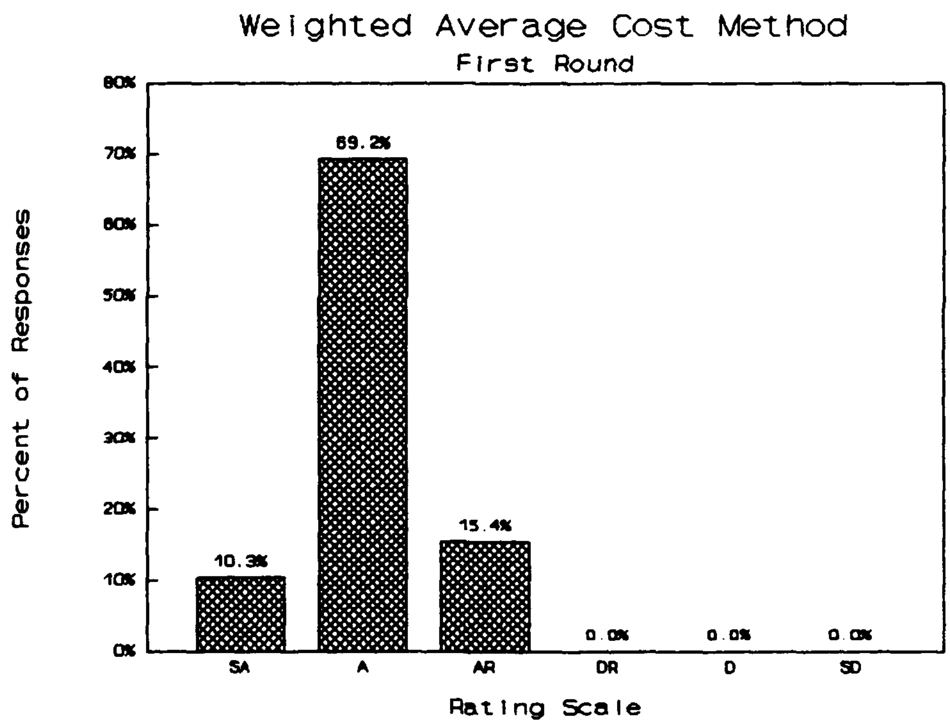
Original Definition:

WEIGHTED AVERAGE COST METHOD

A method of determining the average unit cost of inventory and by implication an aid in determining the cost of goods made, sold, or held for future sale or incorporation into higher level end items. Under this technique, costs are periodically computed by adding the sum of the costs of beginning inventory with the sum of the costs of subsequent purchases and dividing by the total number of units.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Change "goods made, sold, or held..." to "goods made in process, sold to regular customers or held...".

Change "by implication an aid" to "by historical comparisons".

Add to end of definition "Cost values are obtained by multiplying the values by their weights then added together and divided by the sum of the weights."

Synonyms: Unit Cost Comparison Technique.
Antonyms: Specific Identification, Actual Cost Method.

Revised Definition:

WEIGHTED AVERAGE COST METHOD

A method of determining the average unit cost of inventory and by implication an aid in determining the cost of goods made, sold, or held for future sale or incorporation into higher level end items. Under this technique, costs are periodically computed by adding the sum of the costs of beginning inventory with the sum of the costs of subsequent purchases and dividing by the total number of units.

Synonyms: **Unit Cost Comparison Technique**
Antonyms: None

Do you agree with this definition?

-----1------2----------3-----------4----------5----------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

Original Definition:

Allocation

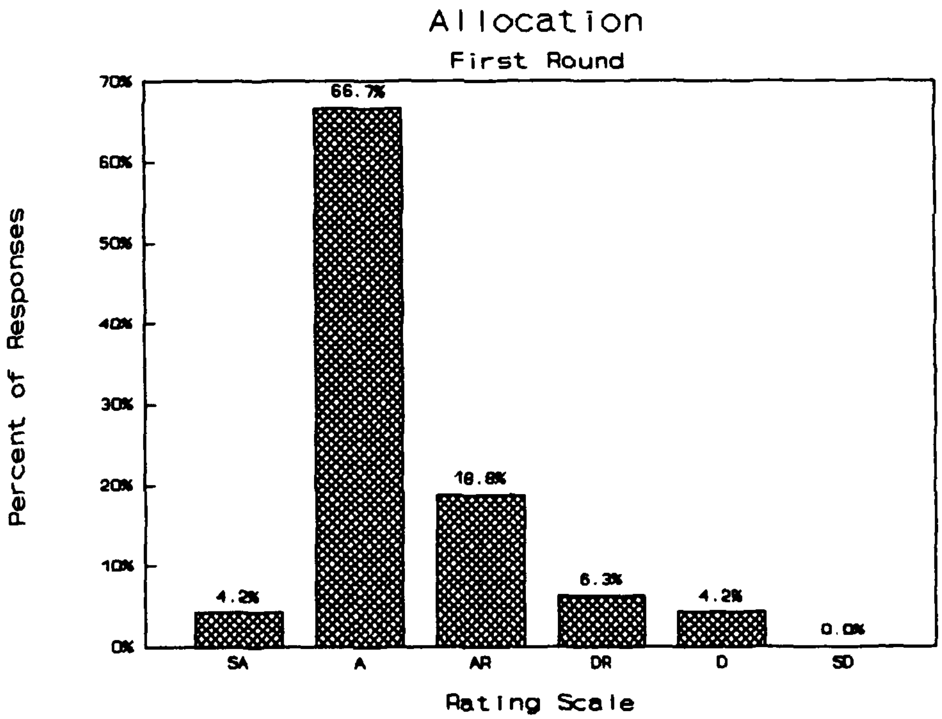
Funding: An amount of money transferred from one agency, bureau or account that is set aside in an appropriation of the various committees having spending responsibilities to carry out the purposes of the parent appropriation or fund.

Within DOD, the money is being transferred from the services to the appropriate MAJCOMS.

Financial: A cost accounting procedure which results in a reasonable distribution of costs among one or more cost objectives (e.g., products, programs, contracts, and activities). This includes both direct assignment of costs and the reassignment of a share from an indirect pool.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Perhaps change reasonable distribution to appropriate distribution.

Funding: An amount of money distributed or assigned by formal action to a particular group or account for a particular use or period of time.

Addition of the meaning of term for resources in the acquisition arena, i.e., allocation of manpower, skills, computer time and memory to tasks.

Add "Represents money that can be obligated." to Funding.

Delete from Funding - "transferred from one agency, bureau or account".

Delete "Within DOD" paragraph.

Change "from" to "to" and "parent appropriation or fund." to "concerned organization" in Funding paragraph.

Add "from DOD to the services" to "Within DOD" paragraph.

Change "transferred" to "earmarked for" in Funding paragraph.

Change "objectives" to "categories" in Financial paragraph. We are not setting objectives when we allocate costs.

Change "one agency, bureau or account" to "entity" in Funding paragraph.

Not sure that funds had to be transferred - allocation could be done by notation or journal entry.

Delete "of the various committees having spending responsibilities" from Funding paragraph.

Change "transferred from the services to the appropriate MAJCOMS." to "flowed down from higher headquarters to the appropriate users."

Change Financial to read "An accounting procedure which assigns costs to an identified usage or purpose."

Change "agency, bureau or account" to "e.g. agency bureau or account" to not limit definition.

Spell out acronym.

Change Financial to read "A cost accounting process of assigning a cost, or group of costs, to one or more cost objectives, in reasonable and realistic proportion to the benefit provided or other equitable relationship.

Delete "Within DOD" paragraph.

Consider the Accounting definition "A systematic distribution or assignment of a total amount among several years, accounts, products, departments or other elements."

Synonyms: Allotment, earmark, assignment, allowance, portion, quota, share allotment, set aside.
Antonyms: Double Counting.

Revised Definition:

Allocation

Funding: An amount of money **in a Government** appropriation **transferred to an agency, bureau or account having spending authority** to carry out the purposes of the that appropriation.

Financial: A cost accounting procedure which results in a reasonable distribution of costs among one or more cost objectives **including** products, programs, contracts, and activities. This includes both direct assignment of costs and the reassignment of a share from an indirect pool.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1------	2---------	3----------	4---------	5---------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____
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SYNONYMS: ____

ANTONYMS: ____

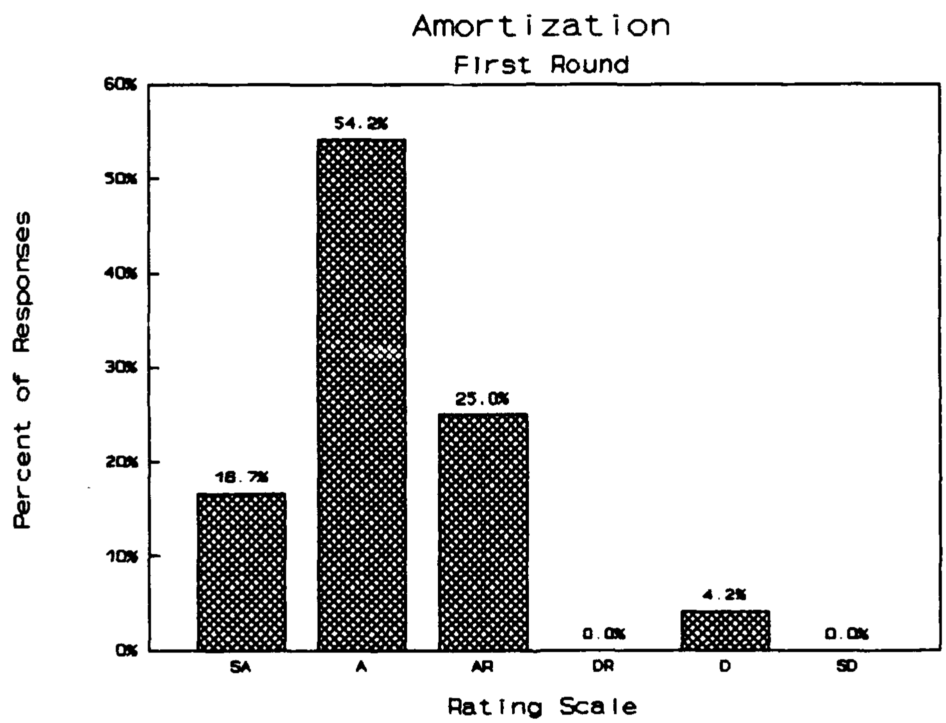
Original Definition:

Amortization

The systematic reduction of an indebtedness or recorded asset value over a specific period of time by periodic payments to a creditor or charges to an expense, in accordance with generally accepted accounting procedures or principles.

Synonyms: Liquidation, Allocation, Writeoff
Antonyms: Direct charge

Survey Results



Comments:

Reference to GAAP not necessary.

Delete allocation as synonym.

Change "asset value" to "asset net value".

Delete direct charge as an antonym.

Change "reduction" to "liquidation".

Change "charges to an expense" to "charges against a capital account".

Change definition to "A system or method which reflects how much of the value of an asset is reduced due to usage or the passage of time."

Add "Amortization is often calculated to occur over a specified period of time."

Add after "asset value", "usually a depreciable capital asset".

Delete "recorded" and "value" from "recorded asset value".

Add after "expense", "account".

Change "reduction of an indebtedness" to "extinguishment of debt".

Synonyms: Depreciation(?), reduction, redemption.
Antonyms: Expense item, Expensed.

Revised Definition:

Amortization

The systematic **liquidation** of an indebtedness or recorded asset value over a specific period of time by periodic payments to a creditor or charges to an expense **account**, in accordance with generally accepted accounting procedures or principles.

Synonyms: Liquidation, Writeoff
Antonyms: **None**

Do you agree with this definition?

-----1------2----------3-----------4----------5----------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____________________

SYNONYMS: ____________________

ANTONYMS: ____________________

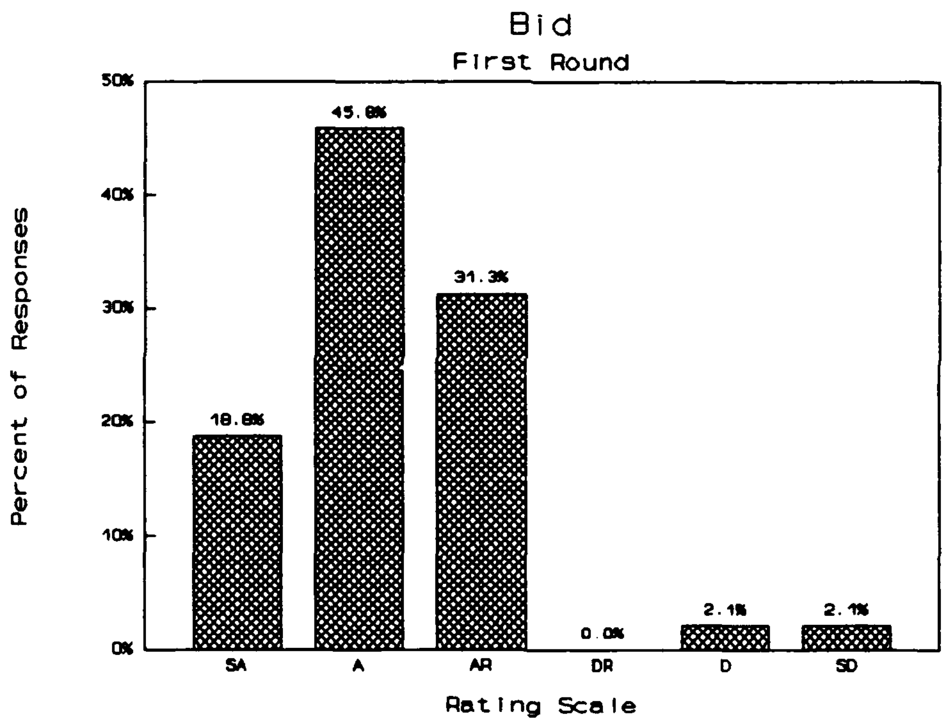
Original Definition:

Bid

An offer to perform a contract by providing labor and or material for a specific price. In federal government contracting, this offer is provided in response to an invitation for bid.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Change "specific price" to "fixed or specific price".

Are "labor and or material" too specific and do they include special test equipment/technical services.

Change "perform a contract by providing labor and or material" to "provide supplies or services for a contract".

Capitalize "Invitation for Bid".

Change "Bid" to "Proposal" - outdated term.

Add to first sentence, "in accordance with specified contract terms.".

Reference legal obligation in definition.

Change "and or material" to "and/or material".

Change "perform" to "form".

Change "invitation for bid" to "a solicitation which invites the submission of such offers.".

Address "sealed bid".

Add to first sentence, "A firm-fixed-price, usually irrevocable, offer to perform ...".

Add to end of definition, "that will not be negotiated.".

Add to end of first sentence, "usually on a firm fixed price basis."

Change second sentence to "To perform the work specified in an invitation for bid (IFB)."

Change first sentence to "An offer to perform the scope of work specified in a contract for a specific price."

Don't limit definition to Government only.

Change first sentence to "An offer by a prospective purchaser to buy goods or services at a stated price, or an offer by a prospective seller to sell his goods or services for a stated price.

Synonyms: Offer.
Antonyms: Request for Proposals (RFP)

Revised Definition:

Bid

An offer to perform a contract by providing **goods or services** for a specific price. In Federal Government contracting, **it is the technical term for an irrevocable offer in response to an Invitation For Bid (IFB).**

Synonyms: **Offer, Proposal.**
Antonyms: None

Do you agree with this definition?

-----1------	2----------	3-----------	4----------	5----------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ______________________________
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SYNONYMS: ______________________________

ANTONYMS: ______________________________

Original Definition:

Commitment

The act by an authorized individual affirming the intent of an agency or company to take or accept a defined action not yet formalized by execution of a contract.

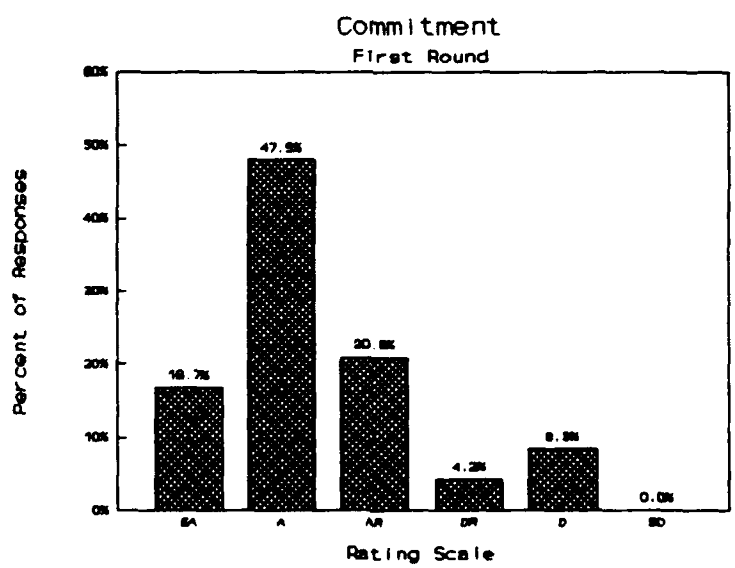
Funding: A firm administrative reservation of funds based upon firm procurement directions, orders, requisitions, certified purchase requests, and budgetary authorizations which set aside certain funds for a particular contract without further recourse to the official responsible for certifying the availability of funds.

Within DOD, reservation of funds are set aside by the appropriate operating division (wing or base) for use on a particular item.

Accounting: The method of accounting for the available balance of an appropriation, fund, or contract authorization whereby commitments are recorded in the accounts as reductions of the available balance.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Change first paragraph to "An engagement by contract involving financial obligation."

Change last paragraph to "Accounting: The total accumulated financial obligation against a contract or other objective at a specific point in time."

Note: A commitment can be either written or unwritten.

Second paragraph add "It indicates intention(s) to incur obligations."

Third paragraph change "use on a particular item." to "something to be bought in the future."

Third paragraph delete "(wing or base)".

Conflict between first and second paragraph - "affirming the intent" and "A firm reservation of funds based on firm procurement directions".

Third paragraph change "division (wing or base)" to "organizations".

Second paragraph change "contract" to "activity". Funding can be set aside for other than contracts, i.e. interagency agreements.

Delete first paragraph - intent is not binding.

First paragraph change "agency or company" to "entity".

Paragraph three delete "reservation of".

Second paragraph change "firm administrative" to "definitive.

Second paragraph change "firm procurement directions" to "procurement directives".

Synonyms: None
Antonyms: None

Revised Definition:

<u>Commitment</u>

The act by an authorized individual affirming the intent of an agency or company to take or accept a defined action not yet formalized by execution of a contract.

Funding: **An administrative** reservation of funds based upon procurement directions, orders, requisitions, certified purchase requests, and budgetary authorizations which set aside certain funds for a particular contract without further recourse to the official responsible for certifying the availability of funds.

Within DOD, funds are set aside by the appropriate operating **organizations** for use on a particular item.

Accounting: The method of accounting for the available balance of an appropriation, fund, or contract authorization whereby commitments are recorded in the accounts as reductions of the available balance.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1------	2---------	-3----------	-4---------	-5---------	-6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____
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SYNONYMS: ____

ANTONYMS: ____

Original Definition:

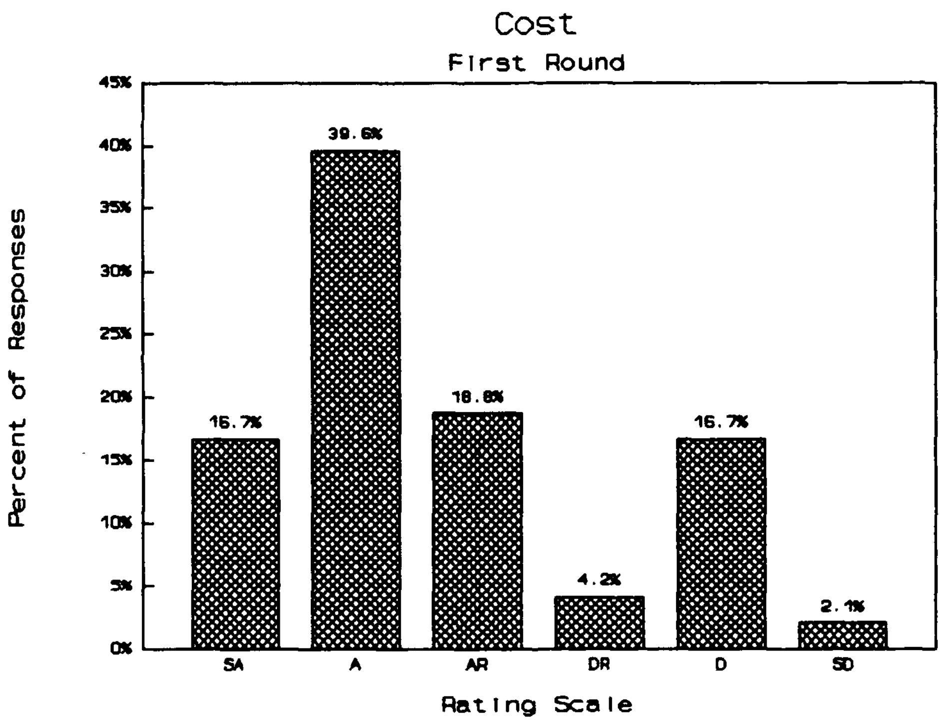
Cost

(1) For the Seller: The amount of money or equivalent incurred for supplies or services exclusive of profit or fee.

(2) For the Buyer: The amount of money or equivalent paid for supplies or services including profit or fee.

Synonyms: Expense, Consideration, Charge, Total Cost
Antonyms: None

Survey Results



Comments:

Change (1) to "The total amount of money or equivalent incurred for the production or purchase of supplies or the performance of services exclusive of profit or fee."

Change (2) to "The amount of money or equivalent paid for supplies or services including the seller's profit or fee, the seller's price."

Remove "*Total Cost*" from Synonyms because total cost may include fee.

Remove "Consideration" from Synonyms.

Add (3) "A direct or indirect charge of a specific or unique element allocated to a particular cost objective."

Consider type of contract - Cost Plus Fixed Fee, cost is separate from profit.

Synonyms: Direct Cost, Indirect Cost, Billed Amount, Actual Cost.
Antonyms: Applicable Credit, Negative Expenditure.

Revised Definitions:

Cost

(1) For the Seller: The amount of money or equivalent incurred for supplies or services exclusive of profit or fee.

(2) For the Buyer: The amount of money or equivalent paid for supplies or services including **the seller's** profit or fee.

Synonyms: Expense, Consideration, Charge, Total Cost
Antonyms: *None*

Do you agree with this definition?

-----1------2---------3----------4---------5---------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

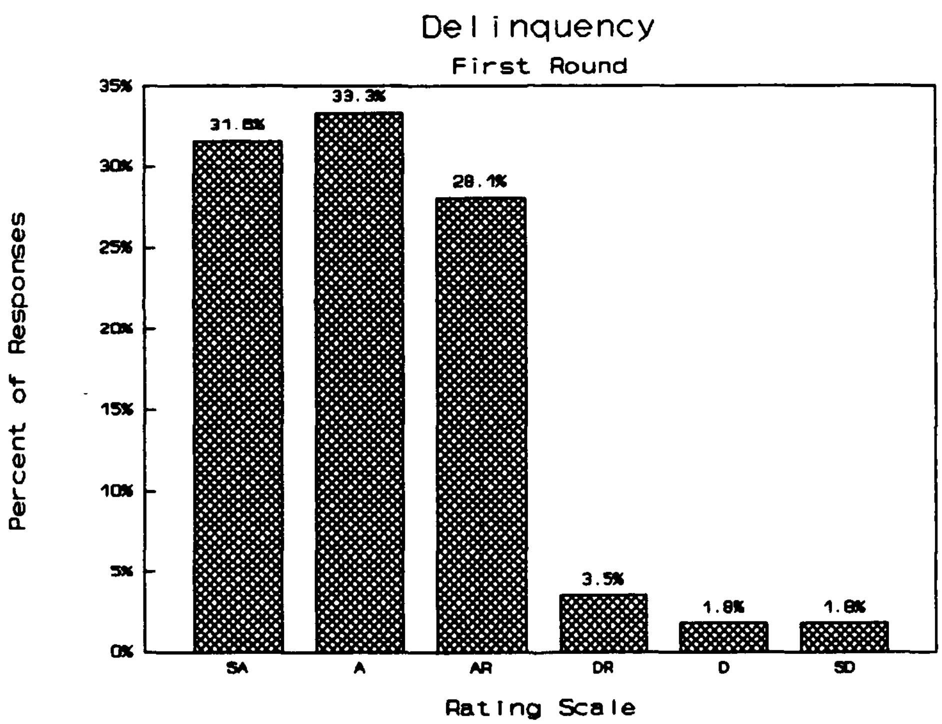
Original Definition:

Delinquency

(1) Failure, omission, or violation of contractual obligation or duty.
(2) The actual failure by the contractor to meet the contract delivery or performance schedule, or the potential failure to do so by failing to maintain required progress in contract performance as required by the contract delivery or performance schedule

Synonyms: Overdue, Tardy, Late
Antonyms: Early, Accelerated, Timely

Survey Results



Comments:

In (2) change "required progress" to "progress".

Change (2) to "The actual failure by a contractor, subcontractor or supplier to meet the contract delivery or performance schedule, or the potential failure to do so by not maintaining progress as required by the contract delivery or performance schedule."

Delete (2).

Is "potential delinquency" a delinquency?

In (2) delete "potential".

In (2) add Government contribution to delinquency by failing to deliver GFE on time.

Change (2) from "meet the contract delivery or performance schedule" to "meet the contract delivery or performance schedule or performance requirements".

Synonyms: Pass Due, Deficient, Substandard Performance, Breach, Noncompliance, In Default, Derelict, Failure, Behind Schedule, Missed Milestone.
Antonyms: Proficient, Standard Performance, Compliant.

Revised Definition:

<u>Delinquency</u>

(1) Failure, omission, or violation of contractual obligation or duty.
(2) The actual failure by the contractor to meet the contract delivery or performance schedule, **performance requirements or** by failing to maintain required progress in contract performance as required by the contract delivery or performance schedule

Synonyms: Overdue, Tardy, Late
Antonyms: Early, Accelerated, Timely

Do you agree with this definition?

-----1------	2----------	3-----------	4----------	5----------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

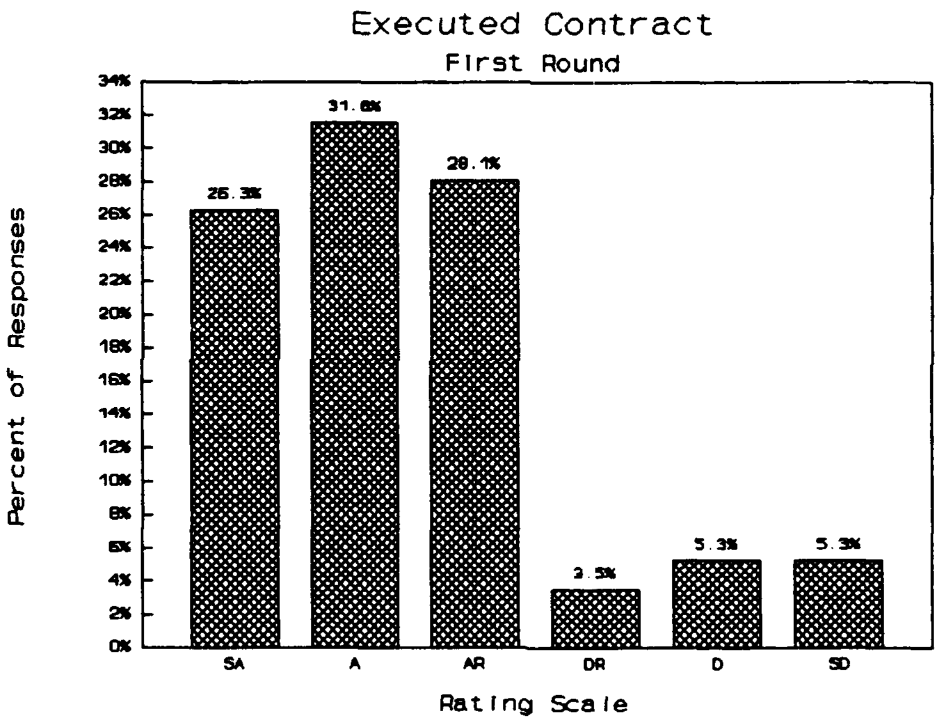
Original Definition:

Executed Contract

A written document which has been signed by both parties and mailed or otherwise furnished to each party, which expresses the requirements, terms, and conditions to be met by each party.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Change "A written document which..." to "A written document in the hands and under control of all parties which..."

Executed or Executory Contract.

Change "and mailed or otherwise" to "and".

Should price/consideration be added?

Change "which expresses" to "which clearly expresses the mutually agreed".

Does definition cover new technology, i.e. EDI, FAX, etc.?

Synonyms: Covenant, Legally Binding Agreement, Definitized Contract, Award, Purchase Agreement, Fully Signed Document.
Antonyms: Ratified, Executory Contract.

Revised Definition:

Executed Contract

A written document which has been signed by both parties and furnished to each party, which expresses the requirements, terms, and conditions to be met by each party.

Synonyms: **Definitized Contract**
Antonyms: None

Do you agree with this definition?

-----1------	2----------	3-----------	4----------	5----------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

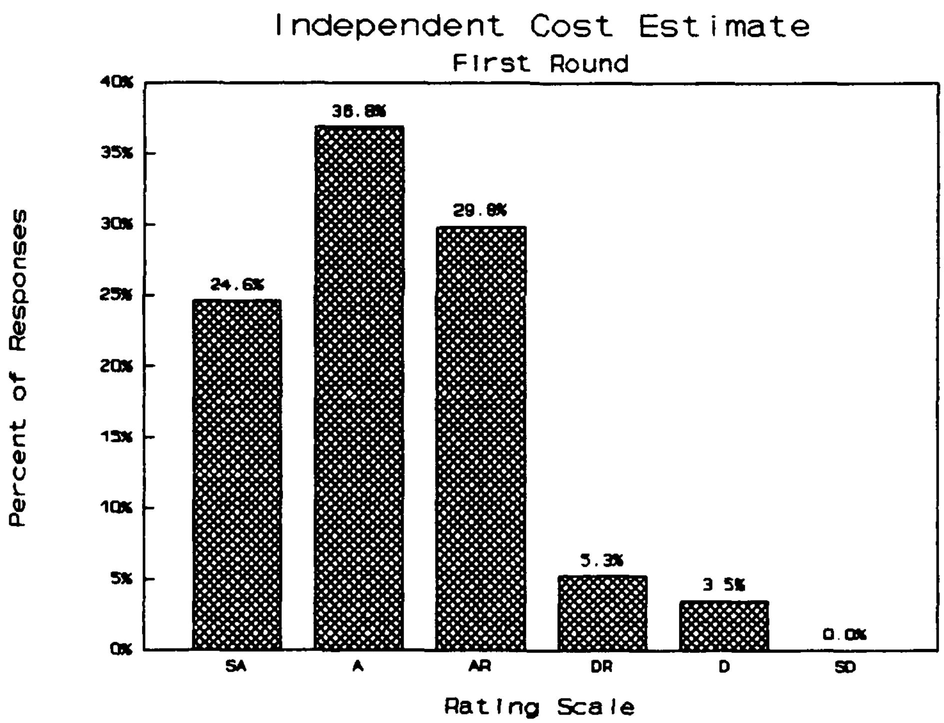
Original Definition:

Independent Cost Estimate

A cost estimate developed outside the normal advocacy channels, independent of any cost information provided by the offeror, used for the purpose of comparing with bids or proposals. Preparation of independent costs estimates generally include representations from the areas of cost analysis, procurement, production management, engineering, and program management.

Synonyms: Independent Government Cost Estimate (IGCE)
Antonyms: None

Survey Results



Comments:

Change "cost analysis" to "cost estimating".

Change "include representations" to "includes input".

Elaborate on advocacy channels.

Add after "developed" - "within the procuring organization".

Add after "proposals" - "and often used in negotiations."

Add after "representations from" - "one or more areas of price/cost analysis".

Put parenthesis around sentence 2.

Synonyms: Should Cost Estimate.
Antonyms: Contractor Prepared Cost Information, Dependent Cost Estimate.

Revised Definition:

Independent Cost Estimate

A cost estimate developed independent of any cost information provided by the offeror, used for the purpose of comparing with bids or proposals. Preparation of independent costs estimates generally includes representations from **one or more of the** areas of cost/**price** analysis, procurement, production management, engineering, and program management.

Synonyms: Independent Government Cost Estimate (IGCE)
Antonyms: **Contractor Prepared Cost Information**

Do you agree with this definition?

-----1------	2---------	3----------	4---------	5---------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____
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SYNONYMS: ____

ANTONYMS: ____

Original Definition:

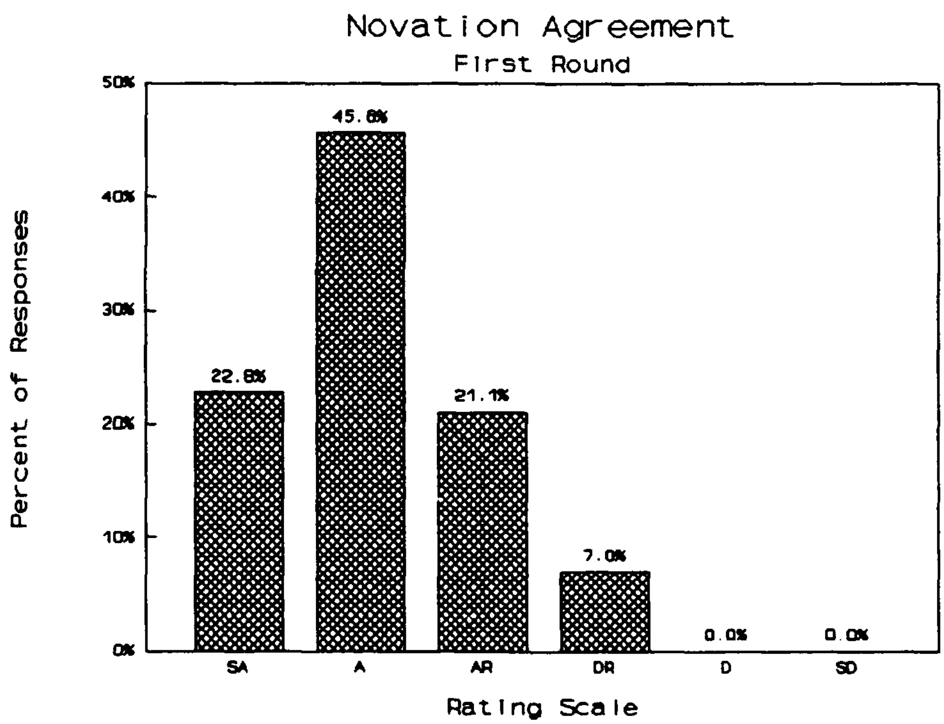
Novation Agreement

A legal instrument, executed by the parties to a contract and a successor in interest, which transfers all obligations and rights under the contract to the successor.

The government may recognize a third party as a successor of a government contract when the third party's interests arises out of the transfer of 1) all the contractor's assets, or 2) the entire portion of the assets involved in the performing a contract.

Synonyms: None
Antonyms: None

Survey Results



Comments:

Change "performing" to "performance of".

Change "under the contract to the successor" to "under the contract of one party to its successor in interest."

Paragraph 2 change "successor" to "successor in interest".

Novation agreements can be made only when a company changes their name only. (?)

Paragraph 2 change "may" to "reserves the right to".

Synonyms: Transfer Agreement, Discharge of Contract, Mutual Rescission, Cancellation, Substituted Contract, Contract Name Change.
Antonyms:

Revised Definition:

Novation Agreement

A legal instrument, executed by the parties to a contract and a successor in interest, which transfers all obligations and rights under the contract **of one party** to the successor **in interest.**

The government **reserves the right to** recognize **or not recognize** a third party as a successor **in interest** of a government contract when the third party's interests arises out of the transfer of 1) all the contractor's assets, or 2) the entire portion of the assets involved in the **performance of** a contract.

Synonyms: None
Antonyms: None

Do you agree with this definition?

-----1------	2---------	3----------	4---------	5---------	6----
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

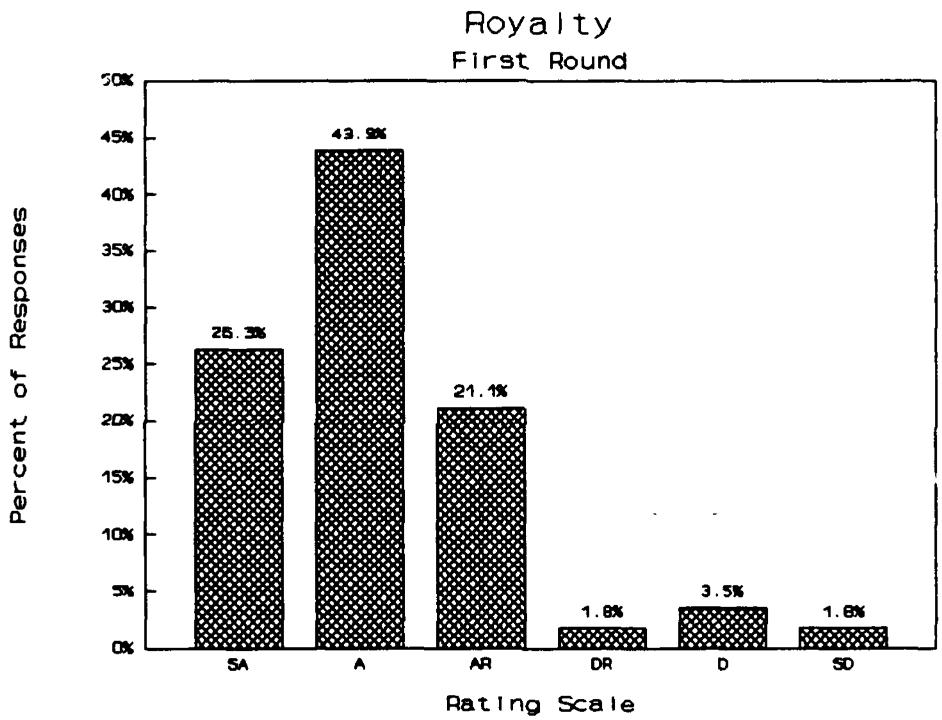
Original Definition:

Royalty

Compensation paid to the owner, vendor or lessor of personal, real, tangible or intangible property for the use of that property. Usually a percentage of the selling price of goods and services, production of which employs the property.

Synonyms: Commission Payment, Use Fee
Antonyms: Royalty Free Use

Survey Results



Comments:

Delete "production of which employs the property".

Sentence 2 not clear.

Do not consider a fee for rental of property to be royalty.

Protected rights of the owner.

Synonyms: Intellectual Property Fee, License Fee, Rent.
Antonyms: No Fee, Rent Free.

Revised Definition:

Royalty

Compensation paid to the owner **or** vendor of personal, real, tangible or intangible property for the use of that property.

Synonyms: Commission Payment, Use Fee, **Intellectual Property Fee.**
Antonyms: Royalty Free Use

Do you agree with this definition?

-----1------2----------3-----------4----------5----------6----

1	2	3	4	5	6
STRONGLY AGREE	AGREE	AGREE W/ RESERVATION	DISAGREE W/ RESERVATION	DISAGREE	STRONGLY DISAGREE

COMMENT: ____

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SYNONYMS: ____

ANTONYMS: ____

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